Typicality in quantum statistical mechanics

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Open-Minded

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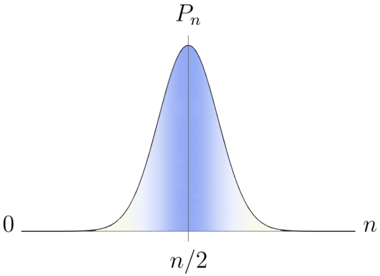
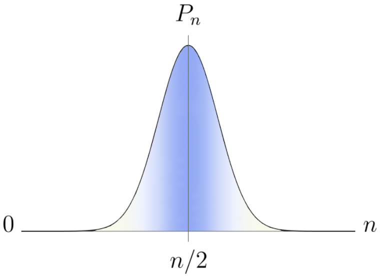
Typicality



Gedankenexperiment:
Tossing a (fair) coin n times

Consider the following result:
$\{1, 1, ...\} \in \{0, 1\}^n$
n small (e.g. $n = 2$): no surprise
n large (e.g. $n = 10^{23}$): highly unlikely or rather impossible

Typicality



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- **The boundary between typical and untypical results is blurred.**
- **Prerequisite: A large sample space (here large n)**

Approach to statistical physics

Usual approach

Macrostate ρ determined by several fixed macroscopic variables (energy, magnetization, ...) and maximization of entropy

time evolution
$\partial_t \rho = -i[H, \rho]$

expectation values
$\langle A \rangle = \text{Tr } \rho A$

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Alternative approach

Microstates in **well defined regions** show very similar properties
Choosing a typical $|\psi\rangle$ will be sufficient

time evolution
$i\partial_t |\psi\rangle = H |\psi\rangle$

expectation values
$\langle A \rangle = \langle \psi | A | \psi \rangle$

Dynamical Quantum Typicality

Finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^N$

Ref.: C. Bartsch and J. Gemmer: Phys. Rev. Lett. **102** 110403 (2009)

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Consider the set of pure states
$M_a = \{|\varphi\rangle \in \mathcal{H}|\, \langle\varphi|\varphi\rangle = 1 \wedge \langle\varphi|A|\varphi\rangle \approx a\}$ then

$$\forall t : |\varphi\rangle\,, |\varphi'\rangle \in M_a : \langle\varphi|B(t)|\varphi\rangle \approx \langle\varphi'|B(t)|\varphi'\rangle$$

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- **This statement is not dependent on the details of the dynamics.**
- **Requirements: A large Hilbert space dimension N and ...**

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Hilbert space averaging

Goal: Find the typical $f(\psi)$ (e.g. expectation value) for random quantum states $|\psi\rangle$

Ref.: J. Gemmer et al: *Quantum Thermodynamics* (2009)

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Hilbert space average

$$\mathrm{HA}\left[f(\psi)\right] := \frac{\int d[\psi] f(\psi)\delta(\langle\psi|\psi\rangle - 1)}{\int d[\psi]\delta(\langle\psi|\psi\rangle - 1)}$$

Confinement to unit sphere by normalization $\langle\psi|\psi\rangle = 1$

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Expectation values

To be more precise

Sampling $|\psi\rangle$ from a **uniform distribution** resp. integration over **Haar measure** of $U(N)$

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⇒ An arbitrary basis $\{|i\rangle\}$ can be chosen.

$$\mathrm{HA}\left[\langle\psi|A|\psi\rangle\right] = \sum_{i,j} A_{ij}\ \mathrm{HA}\left[\psi_i^*\psi_j\right]$$

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- $\mathrm{HA}\left[\sum_i |\psi_i|^2\right] = 1 \Rightarrow \forall i : \mathrm{HA}\left[|\psi_i|^2\right] = \frac{1}{N}$

$$\Rightarrow \mathrm{HA}\left[\langle\psi|A|\psi\rangle\right] = \frac{\mathrm{Tr}\ A}{N}$$

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Changing the distribution

Up to now uniform distribution

Density matrix ρ defines **a new set of states** $|\phi\rangle := \sqrt{N\rho}\,|\psi\rangle$

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$$\mathrm{HA}\left[\langle\phi|\phi\rangle\right] = N\ \mathrm{HA}\left[\langle\psi|\rho|\psi\rangle\right] = \mathrm{Tr}\ \rho = 1$$

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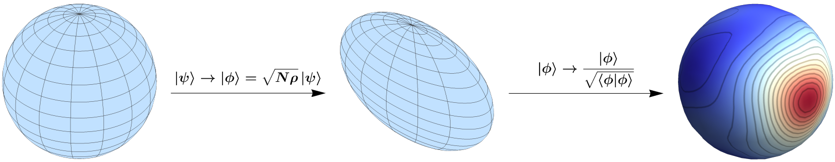
Geometrically speaking



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Validity of Typicality

Hilbert space variance

$$\mathrm{HV}\left[\langle\psi|A|\psi\rangle\right] := \mathrm{HA}\left[\langle\psi|A|\psi\rangle^2\right] - \mathrm{HA}\left[\langle\psi|A|\psi\rangle\right]^2$$
$$= \frac{1}{N+1}\left(\frac{\mathrm{Tr}\ A^2}{N} - \frac{(\mathrm{Tr}\ A)^2}{N^2}\right)$$

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For $|\phi\rangle = \sqrt{N\rho}\,|\psi\rangle$ and **Purity** $P = \mathrm{Tr}\ \rho^2$

$$\mathrm{HV}\left[\langle\phi|\phi\rangle\right] \leq P \qquad \mathrm{HV}\left[\langle\phi|A|\phi\rangle\right] \leq \|A\|^2\, P$$

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$$\text{HV}\,[\langle\phi|A|\phi\rangle] \leq \|A\|^2\,P \qquad \text{HV}\,[\langle\phi|B(t)|\phi\rangle] \leq \|B\|^2\,P$$

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Retrieve Dynamical Quantum Typicality for low Purity

$\Rightarrow$ When is $P \ll 1$?

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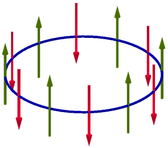
Example: Kicked Ising Chain



n-particle spin-1/2-chain $(N = 2^n)$

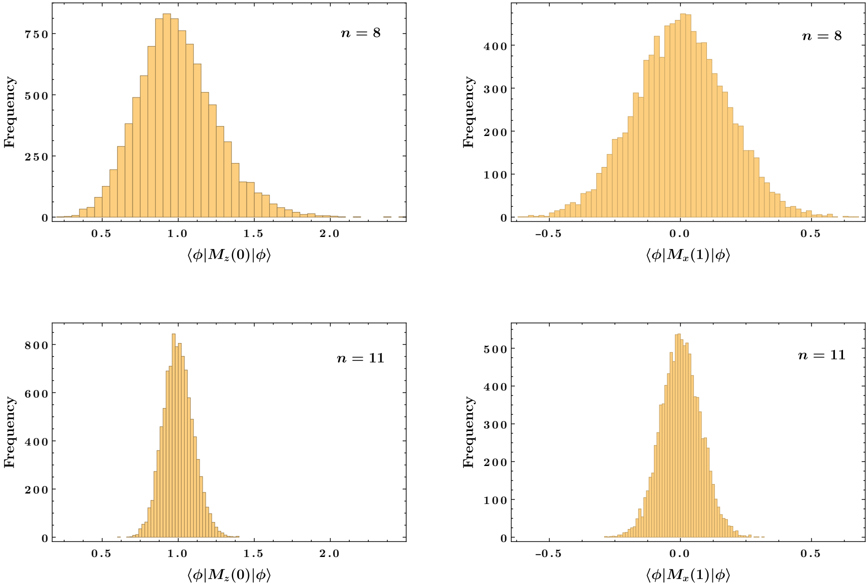
$$H = \sum_{i=1}^{n} J\sigma_i^z\sigma_{i+1}^z + h\,\sigma_i^z + \sum_{i=1}^{n} b\,\sigma_i^x \sum_{\tau=-\infty}^{\infty} \delta(t-\tau)$$

with **discrete time evolution**

$$U = U_I U_K = e^{-iH_I} e^{-iH_K}$$

Magnetization $\quad M_\alpha = \sum_{i=1}^{n} \sigma_i^\alpha \qquad \langle M_\alpha \rangle \in [-n, n]$

Dynamical Quantum Typicality



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Transition probabilities

Now $f(\chi, \psi) = |\langle\chi|A|\psi\rangle|^2$ and $|\chi\rangle$, $|\psi\rangle$ uniformly distributed, A unitary

$$\mathrm{HA}_{\psi,\chi}\left[|\langle\chi|A|\psi\rangle|^2\right] = \mathrm{HA}_{\chi}\left[\frac{\langle\chi|AA^\dagger|\chi\rangle}{N}\right] = \frac{\mathrm{Tr}\ AA^\dagger}{N^2} = \frac{1}{N}$$

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$$\mathsf{HA}_{\psi,\chi}\left[|\langle\chi|A|\psi\rangle|^2\right] = \mathsf{HA}_{\chi}\left[\frac{\langle\chi|AA^\dagger|\chi\rangle}{N}\right] = \frac{\mathrm{Tr}\ AA^\dagger}{N^2} = \frac{1}{N}$$

Impose additional condition: $\langle\chi|\psi\rangle = z$ with $|z| \leq 1$

First leave $|\chi\rangle$ fixed

$$\mathsf{HA}_{\psi}\left[|\langle\chi|A|\psi\rangle|^2\,\delta(\langle\chi|\psi\rangle - z)\right] = \frac{\int d[\psi]\,|\langle\chi|A|\psi\rangle|^2\,\delta(\langle\chi|\psi\rangle - z)\delta(\langle\psi|\psi\rangle - 1)}{\int d[\psi]\delta(\langle\chi|\psi\rangle - z)\delta(\langle\psi|\psi\rangle - 1)}$$

Recipe for integration

$$\mathrm{HA}_\psi\left[\left|\langle\chi|A|\psi\rangle\right|^2\delta(\langle\chi|\psi\rangle - z)\right] = \frac{\int d[\psi]\,\left|\langle\chi|A|\psi\rangle\right|^2\delta(\langle\chi|\psi\rangle - z)\delta(\langle\psi|\psi\rangle - 1)}{\int d[\psi]\delta(\langle\chi|\psi\rangle - z)\delta(\langle\psi|\psi\rangle - 1)}$$

Generating function and Fourier integrals

$$\left|\langle\chi|A|\psi\rangle\right|^2 = \frac{\partial_J}{i}\mathrm{Exp}\ iJ\,\langle\chi|A|\psi\rangle\,\langle\psi|A^\dagger|\chi\rangle\,\Big|_{J=0}$$

$$\delta(x) = \frac{1}{2\pi}\int_{\mathbb{R}} dt\ \mathrm{Exp}\,[itx]$$

$\Rightarrow$ **Arrive at Gaussian integral**

Transition probabilities

For the Hilbert space average- and variance we find

$$\mathrm{HA}_\psi[|\langle\chi|A|\psi\rangle|^2\,\delta(\langle\chi|\psi\rangle - z)] = \frac{1-|z|^2}{N-1}\,\langle\chi|AA^\dagger|\chi\rangle + \frac{N|z|^2-1}{N-1}\,|\langle\chi|A|\chi\rangle|^2$$

$$\mathrm{HV}_\psi\left[|\langle\chi|A|\psi\rangle|^2\,\delta(\langle\chi|\psi\rangle - z)\right] = \lambda_1(N,|z|)\,\langle\chi|AA^\dagger|\chi\rangle^2 + \lambda_2(N,|z|)\,\langle\chi|AA^\dagger|\chi\rangle + \lambda_3(N,|z|)\,|\langle\chi|A|\chi\rangle|^4$$

Special cases

$$\mathrm{HA}_\psi[|\langle\chi|A|\psi\rangle|^2\,\delta(\langle\chi|\psi\rangle - 1)] = |\langle\chi|A|\chi\rangle|^2$$

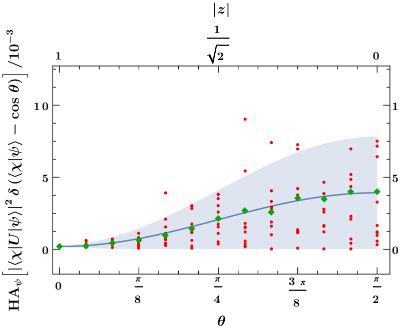
$$\mathrm{HV}_\psi[|\langle\chi|A|\psi\rangle|^2\,\delta(\langle\chi|\psi\rangle - 1)] = 0$$

$$\mathrm{HA}_\psi[|\langle\chi|\psi\rangle|^2\,\delta(\langle\chi|\psi\rangle - z)] = |z|^2$$

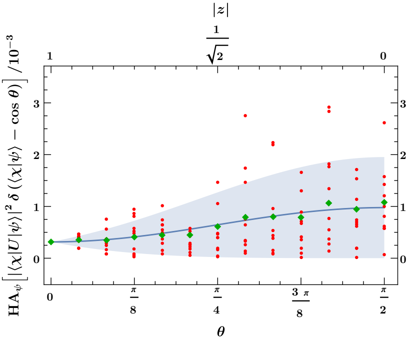
$$\mathrm{HV}_\psi[|\langle\chi|\psi\rangle|^2\,\delta(\langle\chi|\psi\rangle - z)] = 0$$

Transition probabilities

Let $A = U$



$n = 8$

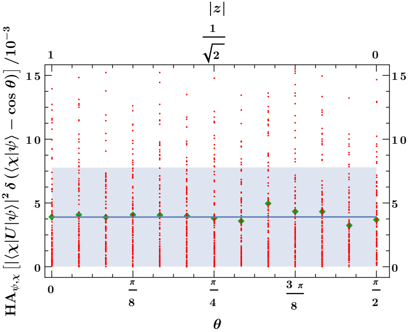


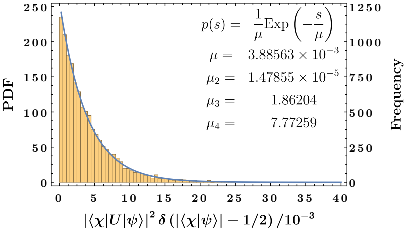
$n = 10$

$$\mathrm{HA}_{\psi}[|\langle\chi|U|\psi\rangle|^2\,\delta(\langle\chi|\psi\rangle - z)] = \frac{1-|z|^2}{N-1} + \frac{N\,|z|^2-1}{N-1}\,|\langle\chi|U|\chi\rangle|^2$$

Transition probabilities

Now average over the remaining state $|\chi\rangle$





$$\mathrm{HA}_{\psi,\chi}\left[|\langle\chi|U|\psi\rangle|^2\,\delta(\langle\chi|\psi\rangle - z)\right] = \frac{N-|z|^2}{N^2-1} + \frac{N|z|^2-1}{N^3-N}\,|\mathrm{Tr}\;U|^2$$

$$\mathrm{HV}_{\psi,\chi}\left[|\langle\chi|U|\psi\rangle|^2\,\delta(\langle\chi|\psi\rangle - z)\right] \approx \text{const.}$$

Spectral Form Factor

With Spectral Form Factor $K(T) = \frac{\left|\mathrm{Tr}\ U^T\right|^2}{N}$

$$\mathrm{HA}_{\psi,\chi}\left[\left|\langle\chi|U|\psi\rangle\right|^2\delta(\langle\chi|\psi\rangle - z)\right] = \frac{N - |z|^2}{N^2 - 1} + \frac{N|z|^2 - 1}{N^2 - 1}K(T)$$

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Dual Operator W **with dim** $W = 2^{T} \times 2^{T}$

$$\mathrm{Tr}\ U^{T} = \mathrm{Tr}\ W^{n}$$

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Dual Operator W **with dim** $W = 2^T \times 2^T$

$$\text{Tr } U^T = \text{Tr } W^n$$

If n large replace **Tr** W^n by the largest eigenvalue $\tilde{\lambda}^n_{\max}(J, b, h)$

$$\text{HA}_{\psi,\chi}\left[|\langle\chi|U|\psi\rangle|^2\,\delta(\langle\chi|\psi\rangle - z)\right] = \frac{N - |z|^2}{N^2 - 1} + \frac{N\,|z|^2 - 1}{N^2 - 1}\left|\tilde{\lambda}^n_{\max}\right|^2$$

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Transition probabilities
Connection to spectral statistics
What happens for non-uniformly distributed $|\psi\rangle$ and $|\chi\rangle$?

Thank you for the attention!