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Ch. Clason and V. H. Nhu

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BOULIGAND–LANDWEBER ITERATION FOR A NON-SMOOTH ILL-POSED PROBLEM

Christian Clason* Vu Huu Nhu*

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Abstract This work is concerned with the iterative regularization of a non-smooth nonlinear ill-posed problem where the forward mapping is merely directionally but not Gâteaux differentiable. Using the Bouligand derivative of the forward mapping, a Landweber-type iteration is derived that converges strongly for exact data as well as in the limit of vanishing data if the iteration is stopped according to the discrepancy principle. The analysis is based on the asymptotic stability of the proposed iteration, which is shown to hold under a generalized tangential cone condition. This is verified for an inverse source problem with a non-smooth Lipschitz continuous nonlinearity. Numerical examples illustrate the convergence of the iterative method.

1 INTRODUCTION

This work is concerned with the (iterative) regularization of inverse problems $F(u) = y$ for a nonlinear parameter-to-state mapping $F : U \to Y$ between two Hilbert spaces U and Y that is compact and directionally but not Gâteaux differentiable. Specifically, we are interested in mappings arising as the solution operator to nonlinear partial differential equations with piecewise continuously differentiable nonlinearities. To fix ideas, let Ω be an open bounded subset of $\mathbb{R}^d$, $d \in \{2, 3\}$, which either has $C^{1,1}$-boundary $\partial\Omega$ or is convex, and consider the non-smooth semilinear equation

$$-\Delta y + y^+ = u \quad \text{in } \Omega, \quad y = 0 \text{ on } \partial\Omega \tag{1.1}$$

with $u \in L^2(\Omega)$ and $y^+(x) := \max(y(x), 0)$ for almost every $x \in \Omega$; see [3]. This equation models the deflection of a stretched thin membrane partially covered by water (see [11]); a similar equation arises in free boundary problems for a confined plasma; see, e.g., [11, 20, 26]. More complicated but related models (where the nonlinearity enters into higher-order terms) can be used to describe problems with sharp phase transitions such as the weak formulation of the two-phase Stefan problem [18, 29].

Our goal is to estimate the source term u in such models from noisy measurements y^δ of the state. For the sake of presentation, in this work we will focus on (1.1), although our results also apply to similar equations with piecewise continuously differentiable nonlinearities in the

*Faculty of Mathematics, University Duisburg-Essen, Thea-Leymann-Strasse 9, 45127 Essen, Germany (christian.clason@uni-due.de, huu.vu@uni-due.de)

potential term. Since solution operators to elliptic equations are usually completely continuous, this problem is ill-posed and has to be regularized. Here we consider iterative regularization methods, which in contrast to Tikhonov regularization allow an efficient regularization parameter choice by the discrepancy principle. Recall that the classical Landweber iteration for a *differentiable* forward mapping $F : U \to Y$ is given by

$$u_{n+1}^\delta = u_n^\delta + w_n F'(u_n^\delta)^* \left(y^\delta - F(u_n^\delta)\right), \quad n \geq 0 \tag{1.2}$$

for a step size $w_n > 0$. For noisy data, the iteration has to be stopped at a stopping index $N = N(\delta) < \infty$ in order to be stable. Assuming that $\|y^\delta - y^\dagger\|_Y \leq \delta$ with $y^\dagger = F(u^\dagger)$ for some $u^\dagger \in U$, it is possible to show that $u_{N(\delta)}^\delta \rightharpoonup u^\dagger$ as $\delta \to 0$, provided that a *tangential cone condition* (which bounds the linearization error by the nonlinear residual) is satisfied at $u^\dagger$; see [8], [10, Chaps. 2, 3], [23, Chap. 10]. Needless to say, if F is not Gâteaux-differentiable, this procedure is not applicable.

However, [22] showed that it is possible to replace the Fréchet derivative $F'(u)$ in (1.2) by another linear operator G_u that is sufficiently close to $F'(u)$ in an appropriate sense; in [14, 15], such an operator was constructed for a class of parameter identification problems for linear elliptic equations. The purpose of this work is to show that G_u can be taken from the Bouligand subdifferential of F, which is defined as the set of limits of Fréchet derivatives in differentiable points as in, e.g., [19, Def. 2.12] or [12, Sec. 1.3] and in our case can be explicitly characterized via the solution of a suitable linearized PDE; cf. (2.8) below. We prove that the resulting *Bouligand–Landweber method* is a convergent regularization method under a corresponding *generalized tangential cone condition*, which is verified for our model problem provided the set of points where the non-smooth nonlinearity is non-differentiable has sufficiently small Lebesgue measure at the exact data $y^\dagger$; cf. Corollary 2.8. The main difficulty here is that the mapping $u \mapsto G_u$ is not continuous (see Example 2.1), which is a critical tool in the classical convergence analysis. We address this by showing that in place of the usual stability of the iterates u_n^δ of (1.2) (which does not hold in our case), it suffices that the iterates are *asymptotically stable* as $n \to \infty$ (cf. Definition 3.1), which follows from our tangential cone condition as well. In particular, our results show that the modified Landweber method of [22] is a regularization under the sole assumption that the family of operators $\{G_u\}_{u \in U}$ is uniformly bounded and satisfies a tangential cone condition.

Let us briefly comment on related literature. Non-smooth inverse problems have attracted immense interest in recent years, although the focus is mainly in the context of non-differentiable regularization methods in Banach spaces; see, e.g., the monographs [21, 25] as well as the references therein. One particular aspect relevant in our context are variational source conditions used to derive convergence rates, which require no explicit assumptions on the regularity of the forward operator and are thus applicable to non-smooth operators as well; see [9]. However, none of the works so far focus on inverse problems for non-differentiable operators. In particular, the construction of G_u in [14, 15] crucially depends on the linearity of the PDE (for a given parameter) and leads to the continuity of the mapping $u \mapsto G_u$, which is in fact required for their analysis. (Hence, their Landweber method is "derivative-free" in the same sense that Krylov methods can be implemented in a "matrix-free" way.) An alternative to iterative regularization is Tikhonov regularization, which for problems of the form (1.1) leads to optimization problems that are known as *mathematical programs with complementarity constraints*, which are challenging both analytically and numerically. Well-posedness and the numerical solution, but not its regularization properties, for the specific example of (1.1) were treated in [3], on which our analysis is based. Similar results for a parabolic version of (1.1) were obtained in [17].

Organization. The paper is organized as follows. After briefly summarizing basic notation, in the next section some auxiliary results relating to the continuity, the complete continuity, and the Bouligand differentiability of F as well as the generalized tangential cone condition are given. The Bouligand–Landweber iteration is the subject in Section 3. In Section 3.1, we show its well-posedness as well as the convergence in the noise-free setting. Section 3.2 is devoted to the asymptotic stability and the regularization property of the our iterative method. Numerical examples illustrating the proposed algorithm are presented in Section 4. The paper ends with an appendix verifying the generalized tangential cone condition for a more general class of non-smooth PDEs involving piecewise differentiable nonlinearities.

Notations. For a Hilbert space X, we denote by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$, respectively, the inner product and the norm on X. For each measurable function u, symbols $\{u < 0\}$, $\{u = 0\}$ and $\{u > 0\}$ stand for the sets of almost every $x \in \Omega$ at which the value of u is negative, zero and positive, respectively. For a measurable set S, we write $|S|$ for the d-dimensional Lebesgue measure of S and denote by $\mathbb{1}_S$ the characteristic function of the set S, i.e., $\mathbb{1}_S(x) = 1$ if $x \in S$ and $\mathbb{1}_S(x) = 0$ if $x \notin S$. The adjoint operator of an operator G will be denoted by G^*.

2 NON-SMOOTH FORWARD OPERATOR

In this section, we give some auxiliary results relating to the well-posedness, Lipschitz continuity and complete continuity of the solution mapping for our model problem and discuss its differentiability properties. We also show that this mapping satisfies the so-called generalized tangential cone condition. These results will be used in the next section to show convergence of our iterative regularization strategy.

2.1 WELL-POSEDNESS AND DIRECTIONAL DIFFERENTIABILITY

Let $\Omega \subset \mathbb{R}^d$, $d \leq 3$, be a bounded domain, which either has $C^{1,1}$ boundary $\partial\Omega$ or is a convex domain. For $u \in L^2(\Omega)$, we define $y_u := y$ as the solution to

$$\begin{cases} -\Delta y + y^+ = u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

From [3], equation (2.1) admits, for each $u \in L^2(\Omega)$, a unique solution y_u belonging to the space

$$\left\{ y \in H_0^1(\Omega) : \Delta y \in L^2(\Omega) \right\}.$$

By the regularity of the domain Ω, [7, Thms. 2.4.2.5 and 3.2.1.2] implies that $y_u \in H^2(\Omega)$ for each $u \in L^2(\Omega)$. Therefore, we have the following well-posedness result.

Proposition 2.1 ([3, Prop. 2.1]). *For all $u \in L^2(\Omega)$, there exists a unique solution $y \in H^2(\Omega) \cap H_0^1(\Omega)$ to (2.1). Moreover, the solution operator $F : u \mapsto y$ associated with (2.1) is well-defined and globally Lipschitz continuous as a function from $L^2(\Omega)$ to $H^2(\Omega) \cap H_0^1(\Omega)$, i.e., there is a constant $C_F > 0$ satisfying*

$$\|F(u) - F(v)\|_{H^2(\Omega)} \leq C_F \|u - v\|_{L^2(\Omega)}, \tag{2.2}$$

$$\|F(u)\|_{H^2(\Omega)} \leq C_F \|u\|_{L^2(\Omega)}, \tag{2.3}$$

for all $u, v \in L^2(\Omega)$.

This implies *a fortiori* that F is continuous from $L^2(\Omega)$ to $L^2(\Omega)$. In our analysis, we will also need the complete continuity of F between these spaces.

Lemma 2.2. *The mapping $F : L^2(\Omega) \to L^2(\Omega)$ is completely continuous, i.e., $u_n \rightharpoonup u$ implies $F(u_n) \to F(u)$.*

Proof. From [3, Cor. 3.8], we obtain that F is weakly continuous from $L^2(\Omega)$ to $H^2(\Omega) \cap H_0^1(\Omega)$. The compact embedding $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ then yields that $F : L^2(\Omega) \to L^2(\Omega)$ is completely continuous. □

We now turn to the differentiability of the solution mapping. We first have that F is (even Hadamard) directionally differentiable.

Proposition 2.3 ([3, Thm. 2.2]). *For any $u \in L^2(\Omega)$ and $h \in L^2(\Omega)$, the mapping $F : L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$ is directionally differentiable, with the directional derivative $F'(u; h)$ given by the solution $\eta \in H^2(\Omega) \cap H_0^1(\Omega)$ to*

$$\begin{cases} -\Delta \eta + \mathbb{1}_{\{y_u = 0\}} \eta^+ + \mathbb{1}_{\{y_u > 0\}} \eta = h & \text{in } \Omega, \\ \eta = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $y_u = F(u)$.

However, F is in general not Gâteaux differentiable.

Proposition 2.4. *Let $u \in L^2(\Omega)$. Then $F : L^2(\Omega) \to L^2(\Omega)$ is Gâteaux differentiable in u if and only if $|\{y_u = 0\}| = 0$.*

Proof. Assume that $|\{y_u = 0\}| = 0$. Then by virtue of [3, Cor. 2.3], $F : L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$ is Gâteaux differentiable in u. Since $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ continuously, F is Gâteaux differentiable in u as a function from $L^2(\Omega)$ to $L^2(\Omega)$. It remains to prove that $F : L^2(\Omega) \to L^2(\Omega)$ is Gâteaux differentiable in u implies $|\{y_u = 0\}| = 0$. Indeed, there exists a bounded operator $S : L^2(\Omega) \to L^2(\Omega)$ such that

$$\frac{F(u + th) - F(u)}{t} \to Sh \quad \text{in} \quad L^2(\Omega) \tag{2.5}$$

as $t \to 0^+$ for any $h \in L^2(\Omega)$. Moreover, the right hand side of (2.5) tends to $F'(u; h)$ in $H^2(\Omega) \cap H_0^1(\Omega)$ and so in $L^2(\Omega)$ whenever $t \to 0^+$. It must hold that $S = F'(u; \cdot)$ and thus $F'(u; h) = -F'(u; -h)$ for any $h \in L^2(\Omega)$. It follows from (2.4) that

$$\mathbb{1}_{\{y_u=0\}} |F'(u; h)| = \mathbb{1}_{\{y_u=0\}} (F'(u; h))^+ + \mathbb{1}_{\{y_u=0\}} (-F'(u; h))^+ = 0 \tag{2.6}$$

for all $h \in L^2(\Omega)$. By [3, Lem. A.1], there exist a function $\psi \in C^\infty(\mathbb{R}^d)$ satisfying $\psi > 0$ in Ω and $\psi = 0$ in $\mathbb{R}^d \backslash \Omega$. Let choose function $h \in L^2(\Omega)$ as follows

$$h := -\Delta\psi + \mathbb{1}_{\{y_u \geq 0\}}\psi.$$

We then have $F'(u;h) = \psi$. Plugging this into (2.6) yields $\mathbb{1}_{\{y_u=0\}}\psi = 0$. Consequently, we have $|\{y_u = 0\}| = 0$ as claimed. □

The directional derivative is difficult to exploit algorithmically. A more convenient object can be constructed using the Bouligand subdifferential, which also arises in the definition of the Clarke subdifferential [4] (as the convex hull of the Bouligand subdifferential) and is used in the construction of semismooth Newton methods [1, 28] (as a set of candidates for slant or Newton derivatives). We first define the set of *Gâteaux points* of F as

$$D := \{v \in L^2(\Omega) : F : L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega) \text{ is Gâteaux differentiable in } v\}.$$

The (strong-strong) *Bouligand subdifferential* at $u \in L^2(\Omega)$ is then defined as

$$\begin{aligned}\partial_B F(u) := \{G_u \in \mathcal{L}(L^2(\Omega), H^2(\Omega) \cap H_0^1(\Omega)) &: \text{there exists } \{u_n\} \subset D \text{ such that} \\ u_n \to u \text{ in } L^2(\Omega) \text{ and } F'(u_n; h) &\to G_u\, h \text{ in } H^2(\Omega) \cap H_0^1(\Omega) \text{ for all } h \in L^2(\Omega)\}.\end{aligned}$$

(By replacing one or both convergences with the corresponding weak convergence, we arrive at different variants of the Bouligand subdifferential; for our purposes, however, the strong notion suffices. See [3, Sec. 3.1] for the precise definitions and the relations between them.) From the definition, it follows that G_u is uniformly bounded for all $u \in L^2(\Omega)$ and that if F is Gâteaux differentiable in u, then $G'(u) \in \partial_B F(u)$; see [3, Lem. 3.3]. In particular, we deduce that there exist constants L and $\hat{L}$ satisfying

$$\|G_u\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} \leq L, \quad \|G_u\|_{\mathcal{L}(L^2(\Omega), H^2(\Omega)\cap H_0^1(\Omega))} \leq \hat{L}. \tag{2.7}$$

We can give a convenient characterization of a specific Bouligand subderivative of F.

Proposition 2.5 ([3, Prop. 3.16]). *Given $u \in L^2(\Omega)$, let $G_u : L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ be the solution operator mapping $h \in L^2(\Omega)$ to the unique solution $\eta \in H^2(\Omega) \cap H_0^1(\Omega)$ to*

$$\begin{cases} -\Delta\eta + \mathbb{1}_{\{y_u > 0\}}\eta = h & \text{in } \Omega, \\ \eta = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

where $y_u := F(u)$. Then $G_u \in \partial_B F(u)$.

We refer to [3, Thm. 3.18] for a precise characterization of the full Bouligand subdifferential. Clearly, G_u is a self-adjoint operator when considered acting from $L^2(\Omega)$ to $L^2(\Omega)$. Furthermore, for this specific choice of the subderivative, we can derive an L^p version of the estimates (2.7) which will be needed in the following.

Lemma 2.6. *Let $\frac{d}{2} < p \leq 2$. Then there exists a constant $L_p > 0$ such that*

$$\|G_u\|_{\mathcal{L}(L^p(\Omega),C(\overline{\Omega}))} \leq L_p \quad \text{for all } u \in U. \tag{2.9}$$

Furthermore, if $p = 2$, there exists a constant $\tilde{L} > 0$ such that

$$\|G_u\|_{\mathcal{L}(L^2(\Omega),H^2(\Omega))} \leq \tilde{L} \quad \text{for all } u \in U. \tag{2.10}$$

Proof. Let $h \in L^p(\Omega)$ with $\frac{d}{2} < p \leq 2$ and $u \in U$ be arbitrary. From Proposition 2.5, we have that $\eta = G_u h$ satisfies

$$\begin{cases} -\Delta\eta + a\eta = h & \text{in } \Omega, \\ \eta = 0 & \text{on } \partial\Omega. \end{cases}$$

for some $a \in L^\infty(\Omega)$ with $0 \leq a(x) \leq 1$ for a.e. $x \in \Omega$. Stampacchia's theorem [2, Thm. 12.4] and [27, Thm. 4.7] thus ensure that $\eta \in C(\overline{\Omega}) \cap H_0^1(\Omega)$ and satisfies

$$\|\eta\|_{C(\overline{\Omega})} \leq L_p\|h\|_{L^p(\Omega)} \quad \text{for } h \in L^p(\Omega) \tag{2.11}$$

for some constant L_p independent of a and h, i.e., (2.9).

For the a priori estimate (2.10), we multiply both sides of the equation with η and integrate over Ω to obtain

$$\|\nabla\eta\|_{L^2(\Omega)} \leq \|h\|_{L^2(\Omega)}.$$

From this and Poincaré's inequality, it follows that

$$\|\eta\|_{L^2(\Omega)} \leq K\|h\|_{L^2(\Omega)}$$

with constant K independent of a and h. We then have

$$\|\Delta\eta\|_{L^2(\Omega)} = \|h - a\eta\|_{L^2(\Omega)} \leq \|h\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)} \leq (1+K)\|h\|_{L^2(\Omega)}.$$

Consequently,

$$\|\eta\|_{H^2(\Omega)} = \left(\|\eta\|^2_{L^2(\Omega)} + \|\nabla\eta\|^2_{L^2(\Omega)} + \|\Delta\eta\|^2_{L^2(\Omega)}\right)^{1/2} \leq \left(1 + K^2 + (1+K)^2\right)^{1/2} \|h\|_{L^2(\Omega)}.$$

By setting $\tilde{L} := \left(1 + K^2 + (1+K)^2\right)^{1/2}$, we obtain

$$\|\eta\|_{H^2(\Omega)} \leq \tilde{L}\|h\|_{L^2(\Omega)} \quad \text{for } h \in L^2(\Omega) \tag{2.12}$$

and hence (2.10). □

Finally, the following example shows that the mapping $u \mapsto G_u h$ is in general not continuous, which is the main difficulty in showing convergence of the Bouligand–Landweber method.

Example 2.1. Let Ω be a unit ball in $\mathbb{R}^2$. For each $\varepsilon > 0$, we set

$$u_\varepsilon(x) := \varepsilon\left(5 - x_1^2 - x_2^2\right).$$

Then u_ε tends to $\bar{u} := 0$ as $\varepsilon \to 0^+$. Furthermore, we have $y_\varepsilon(x) = F(u_\varepsilon)(x) = \varepsilon(1 - x_1^2 - x_2^2) > 0$ for all $x = (x_1, x_2) \in \Omega$. It follows that $\mathbb{1}_{\{y_\varepsilon > 0\}}(x) = 1$ almost every where in Ω, and hence $G_{u_\varepsilon} \equiv G$ for the operator $G : L^2(\Omega) \to L^2(\Omega)$ defined by $z := Gh$ being a unique solution to

$$\begin{cases} -\Delta z + z = h & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

On the other hand, $\bar{z} := G_{\bar{u}}h$ satisfies

$$\begin{cases} -\Delta \bar{z} = h & \text{in } \Omega, \\ \bar{z} = 0 & \text{on } \partial\Omega. \end{cases}$$

for any $h \in L^2(\Omega)$. We thus have $z \neq \bar{z}$ whenever $h \neq 0$. Therefore, if $h \neq 0$,

$$G_{u_\varepsilon}h \nrightarrow G_{\bar{u}}h \quad \text{as} \quad \varepsilon \to 0^+.$$

2.2 GENERALIZED TANGENTIAL CONE CONDITION

We now verify that the solution mapping for our example satisfies a generalized tangential cone condition with G_u in place of the non-existent Fréchet derivative $F'(u)$. We begin with a crucial lemma deriving a "pointwise" tangential cone condition.

Lemma 2.7. *Let $u, \hat{u} \in L^2(\Omega)$ and $\frac{d}{2} < p < 2$. Then, one has*

$$\|F(\hat{u}) - F(u) - G_u(\hat{u} - u)\|_{L^2(\Omega)} \leq L_p|\Omega|^{1/2}M(u,\hat{u})^{1/p'}\|F(\hat{u}) - F(u)\|_{L^2(\Omega)}$$

where $p' = \frac{2p}{2-p}$, L_p is given in Lemma 2.6, and

$$M(u,\hat{u}) := |\{y_u \leq 0, y_{\hat{u}} > 0\} \cup \{y_u > 0, y_{\hat{u}} \leq 0\}|.$$

Proof. Let us set $y := y_u$, $\hat{y} := y_{\hat{u}}$, $\zeta := G_u(\hat{u} - u)$, and $\omega := \hat{y} - y - \zeta$. We then have from the definitions that

$$\begin{aligned} -\Delta\hat{y} + \hat{y}^+ &= \hat{u}, \\ -\Delta y + y^+ &= u, \\ -\Delta\zeta + \mathbb{1}_{\{y>0\}}\zeta &= \hat{u} - u. \end{aligned}$$

This implies that

$$-\Delta\omega + \mathbb{1}_{\{y>0\}}\omega = \left(\mathbb{1}_{\{y>0\}} - \mathbb{1}_{\{\hat{y}>0\}}\right)\hat{y}.$$

By simple computation, we have

$$a := \left(\mathbb{1}_{\{y>0\}} - \mathbb{1}_{\{\hat{y}>0\}}\right)\hat{y} = \left(\mathbb{1}_{\{y>0,\hat{y}\leq 0\}} - \mathbb{1}_{\{y\leq 0,\hat{y}>0\}}\right)\hat{y}$$

and so

$$0 \geq a(x) \geq \left(\mathbb{1}_{\{y>0,\hat{y}\leq 0\}} - \mathbb{1}_{\{y\leq 0,\hat{y}>0\}}\right)(\hat{y} - y).$$

Consequently,

$$|a(x)| \leq |e(x)||\hat{y}(x) - y(x)| \quad \text{for a.e. } x \in \Omega$$

with

$$e := \left(\mathbb{1}_{\{y>0,\hat{y}\leq 0\}} - \mathbb{1}_{\{y\leq 0,\hat{y}>0\}}\right).$$

From this, Lemma 2.6, and the Hölder inequality, we get

$$\begin{aligned}\|\omega\|_{C(\overline{\Omega})} &\leq L_p\|a\|_{L^p(\Omega)}\\ &\leq L_p\|\hat{y} - y\|_{L^2(\Omega)}\|e\|_{L^{p'}(\Omega)}\\ &\leq L_p M(u,\hat{u})^{1/p'}\|\hat{y} - y\|_{L^2(\Omega)}.\end{aligned}$$

This together with the inequality $\|\omega\|_{L^2(\Omega)} \leq \|\omega\|_{C(\overline{\Omega})}|\Omega|^{1/2}$ implies the desired inequality. □

The following result is immediate consequence of Lemma 2.7, which verifies a "generalized tangential cone condition", is close to the classical tangential cone condition (see [8, 16, 22]), and will be crucial in the analysis.

Corollary 2.8. *Let μ be a positive number and assume that*

$$L_p|\Omega|^{1/2}\left(2|\{y^\dagger = 0\}|\right)^{1/p'} < \frac{\mu}{2} \tag{2.13}$$

with $p' := \frac{2p}{2-p}$ and L_p is given in Lemma 2.6. Then there exists $\rho > 0$ such that

$$\|F(\hat{u}) - F(u) - G_u(\hat{u} - u)\|_{L^2(\Omega)} \leq \mu\|F(\hat{u}) - F(u)\|_{L^2(\Omega)} \tag{GTCC}$$

for all $u, \hat{u} \in B_{L^2(\Omega)}(u^\dagger, \rho)$.

Proof. The Lipschitz continuity of F and the embedding $C(\overline{\Omega}) \hookrightarrow H^2(\Omega)$ ensure that

$$\|y^\dagger - y_u\|_{C(\overline{\Omega})} \leq C_\infty\|y^\dagger - y_u\|_{H^2(\Omega)} \leq C_\infty C_F\|u^\dagger - u\|_{L^2(\Omega)} < C_\infty C_F \rho =: \varepsilon$$

for all $u \in B_{L^2(\Omega)}(u^\dagger, \rho)$. Then, for any $u \in B_{L^2(\Omega)}(u^\dagger, \rho)$, it follows that

$$-\varepsilon + y_u(x) < y^\dagger < \varepsilon + y_u(x)$$

for all $x \in \bar{\Omega}$. This implies for any $u, \hat{u} \in B_{L^2(\Omega)}(u^\dagger, \rho)$ that

$$\begin{aligned}\{y_u > 0, y_{\hat{u}} \leq 0\} &\subset \{-\varepsilon < y^\dagger < \varepsilon\},\\ \{y_u \leq 0, y_{\hat{u}} > 0\} &\subset \{-\varepsilon < y^\dagger < \varepsilon\}\end{aligned}$$

and thus

$$M(u,\hat{u}) \leq 2|\{0 \leq |y^\dagger| < \varepsilon\}|.$$

From condition (2.13), we have

$$\lim_{\varepsilon \to 0^+} L_p |\Omega|^{1/2} \left(2|\{0 \le |y^\dagger| < \varepsilon\}| \right)^{1/p'} < \frac{\mu}{2}.$$

Note that $\varepsilon \to 0^+$ as $\rho \to 0^+$. Choosing $\rho > 0$ small enough such that

$$L_p |\Omega|^{1/2} \left(2|\{0 \le |y^\dagger| < \varepsilon\}| \right)^{1/p'} \le \mu$$

yields that

$$\|F(\hat{u}) - F(u) - G_u(\hat{u} - u)\|_{L^2(\Omega)} \le \mu \|F(\hat{u}) - F(u)\|_{L^2(\Omega)}$$

for all $u, \hat{u} \in B_{L^2(\Omega)}(u^\dagger, \rho)$. □

The condition (2.13) is related to – but weaker than – the active set condition introduced in [30, 31] in order to derive strong convergence rates for the Tikhonov regularization of singular and nonsmooth optimal control problems. We stress that the condition (2.13) does not require that F is differentiable at the exact solution $u^\dagger$.

Remark 2.9. The results of this section – and hence of this work – can be extended to the case of piecewise continuously differentiable nonlnonlinearities i.e., to forward operators given as the solution mapping to

$$\begin{cases} Ay + f(y) = u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.14}$$

where A is a second-order strongly uniformly elliptic operator and f is a piecewise continuously differentiable and non-decreasing; see Appendix A.

3 BOULIGAND–LANDWEBER METHOD

We now turn to the convergence of the Bouligand–Landweber method for the inverse problem

$$F(u) = y^\dagger \tag{3.1}$$

for some $y^\dagger \in R(F)$, i.e., there exists a $u^\dagger \in U$ with $F(u^\dagger) = y^\dagger$. Let ρ_0 be a positive number. We assume that the mapping $F : U \to Y$ between the Hilbert spaces U and Y satisfies the following conditions.

(A1) F is completely continuous.

(A2) For each $u \in B_U(u^\dagger, \rho_0)$, there exists a linear operator $G_u : U \to Y$ such that $\|G_u\|_{\mathcal{L}(U,Y)} \le L$ for some constant L independent of u.

(A3) There exist constants $\rho \in (0, \rho_0]$ and $\mu \in [0, 1)$ such that (GTCC) holds for all $u, \hat{u}$ in $B_U(u^\dagger, \rho)$.

(A4) There exists a Banach space Z such that Z is compactly embedded in U and contains the ranges of $G_u^* : Y \to U$ for all $u \in U$. Moreover, there is a constant $\hat{L}$ such that $\|G_u^*\|_{\mathcal{L}(Y,Z)} \le \hat{L}$ for all $u \in B_U(u^\dagger, \rho)$.

Note that we do not require the continuity of the mapping $u \mapsto G_u$. Furthermore, Assumption (A2) is always satisfied for Bouligand differentiable mappings. If $U = Y = L^2(\Omega)$ and F is the solution operator to (1.1), then Assumptions (A1) to (A4) hold with G_u defined as in Proposition 2.5 and $Z = H^2(\Omega) \cap H_0^1(\Omega)$, provided that condition (2.13) is valid; see Lemma 2.2, (2.7), Corollary 2.8, and Lemma 2.6, respectively. We also note that in this case F is injective, i.e., $u^\dagger$ is the unique solution to (3.1).

Let $S(u^\dagger, \rho)$ stand for the set of all solutions in $B_U(u^\dagger, \rho)$ of (3.1), that is,

$$S(u^\dagger, \rho) := \left\{u \in B_U(u^\dagger, \rho) : F(u) = y^\dagger\right\}.$$

Obviously, $u^\dagger$ belongs to $S(u^\dagger, \rho)$.

Let now $y^\delta \in Y$ with $\|y^\delta - y^\dagger\|_Y \le \delta$. The *Bouligand–Landweber iteration* is then given by

$$u_{n+1}^\delta = u_n^\delta + w_n (G_{u_n^\delta})^* \left(y^\delta - F(u_n^\delta)\right), \quad n = 0, 1, 2, \dots, \tag{3.2}$$

for the initial guess $u_0^\delta := u_0$ and the step sizes $w_n > 0$. The iteration is stopped after $N(\delta)$ steps according to the *discrepancy principle*, i.e.,

$$\|y^\delta - F(u_{N(\delta)}^\delta)\|_Y \le \tau\delta < \|y^\delta - F(u_n^\delta)\|_Y, \quad 0 \le n < N(\delta) \tag{3.3}$$

with $\tau > 1$ being a positive constant.

Remark 3.1. Let $U = Y = L^2(\Omega)$ and that F be the solution operator to (1.1). If $|\{y_n^\delta = 0\}| = 0$ with $y_n^\delta := F(u_n^\delta)$ for some $n \in \mathbb{N}$, we obtain from Proposition 2.4 that F is Gâteaux differentiable in u_n^δ and that $G_{u_n^\delta} = F'(u_n^\delta)$ and hence that the corresponding Bouligand–Landweber step (3.2) coincides with the classical Landweber step (1.2).

3.1 WELL-POSEDNESS AND CONVERGENCE

We first show the well-posedness of (3.2). The proof of the following lemma is similar to the one in [8, Prop. 2.2] with some modifications.

Lemma 3.2. *Assume that Assumptions (A2) and (A3) are fulfilled. Let us choose positive numbers τ, λ, Λ such that*

$$\lambda \le \Lambda, \quad \frac{2(\mu+1)}{\tau} - (2 - 2\mu - \Lambda L^2) < 0. \tag{3.4}$$

Then, for any initial guess $u_0 \in B_U(u^\dagger, \rho)$ and the step sizes $w_n \in [\lambda, \Lambda]$, the sequence $\{u_n^\delta\}_{0 \le n \le N(\delta)}$ generated by (3.2) with the stopping index $N(\delta)$ defined by the discrepancy principle (3.3) satisfies the following assertions:

(i) the stopping index is finite, i.e., $N(\delta) < \infty$;

(ii) $\|u_{n+1}^\delta - \tilde u\|_U < \|u_n^\delta - \tilde u\|_U$ *for all* $0 \le n \le N(\delta) - 1$ *and for any* $\tilde u \in S(u^\dagger, \rho)$*. Consequently,* $u_n^\delta \in B_U(u^\dagger, \rho)$ *for all* $0 \le n \le N(\delta)$*.*

Proof. Let $\tilde u$ be an arbitrary element of $S(u^\dagger, \rho)$. We have

$$\begin{aligned}
&\|u_{n+1}^\delta - \tilde u\|_U^2 - \|u_n^\delta - \tilde u\|_U^2 \\
&\qquad = 2\left(u_n^\delta - \tilde u, u_{n+1}^\delta - u_n^\delta\right)_U + \|u_{n+1}^\delta - u_n^\delta\|_U^2 \\
&\qquad = 2w_n\left(G_{u_n^\delta}(u_n^\delta - \tilde u), y^\delta - F(u_n^\delta)\right)_Y + \|u_{n+1}^\delta - u_n^\delta\|_U^2 \\
&\qquad = 2w_n\left(F(\tilde u) - F(u_n^\delta) - G_{u_n^\delta}(\tilde u - u_n^\delta), y^\delta - F(u_n^\delta)\right)_Y \\
&\qquad\quad - 2w_n\left(F(\tilde u) - F(u_n^\delta), y^\delta - F(u_n^\delta)\right)_Y + \|u_{n+1}^\delta - u_n^\delta\|_U^2 \\
&\qquad = 2w_n\left(F(\tilde u) - F(u_n^\delta) - G_{u_n^\delta}(\tilde u - u_n^\delta), y^\delta - F(u_n^\delta)\right)_Y - 2w_n\|y^\delta - F(u_n^\delta)\|_Y^2 \\
&\qquad\quad - 2w_n\left(y^\dagger - y^\delta, y^\delta - F(u_n^\delta)\right)_Y + w_n^2\left\|(G_{u_n^\delta})^*\left(y^\delta - F(u_n^\delta)\right)\right\|_U^2,
\end{aligned}$$

which together with (GTCC) implies that

$$\begin{aligned}
(3.5)\quad &\|u_{n+1}^\delta - \tilde u\|_U^2 - \|u_n^\delta - \tilde u\|_U^2 \\
&\le 2w_n\mu\|y^\dagger - F(u_n^\delta)\|_Y\|y^\delta - F(u_n^\delta)\|_Y - 2w_n\|y^\delta - F(u_n^\delta)\|_Y^2 \\
&\quad + 2w_n\delta\|y^\delta - F(u_n^\delta)\|_Y + L^2w_n^2\|y^\delta - F(u_n^\delta)\|_Y^2 \\
&= w_n\|y^\delta - F(u_n^\delta)\|_Y\left[2\mu\|y^\dagger - F(u_n^\delta)\|_Y - (2 - L^2w_n)\|y^\delta - F(u_n^\delta)\|_Y + 2\delta\right].
\end{aligned}$$

Here we have used the fact that $\|G_u^*\|_{\mathcal{L}(Y,U)} = \|G_u\|_{\mathcal{L}(U,Y)}$ and the uniform bound from Assumption (A2). From the discrepancy principle (3.3), one has

$$(3.6)\qquad \delta < \frac{1}{\tau}\|y^\delta - F(u_n^\delta)\|_Y \quad \text{for all } 0 \le n < N(\delta)$$

and so

$$\begin{aligned}
\|y^\dagger - F(u_n^\delta)\|_Y &\le \delta + \|y^\delta - F(u_n^\delta)\|_Y \\
&< \left(\frac{1}{\tau} + 1\right)\|y^\delta - F(u_n^\delta)\|_Y
\end{aligned}$$

for all $0 \le n < N(\delta)$. This together with (3.5) and (3.6) implies for all $0 \le n < N(\delta)$ that

$$\begin{aligned}
(3.7)\quad \|u_{n+1}^\delta - \tilde u\|_U^2 - \|u_n^\delta - \tilde u\|_U^2 &< w_n\|y^\delta - F(u_n^\delta)\|_Y^2\left[2\mu\left(\frac{1}{\tau} + 1\right) - (2 - L^2w_n) + \frac{2}{\tau}\right] \\
&\le w_n\left(\frac{2(\mu+1)}{\tau} - (2 - 2\mu - \Lambda L^2)\right)\|y^\delta - F(u_n^\delta)\|_Y^2 \\
&\le \lambda\left(\frac{2(\mu+1)}{\tau} - (2 - 2\mu - \Lambda L^2)\right)\|y^\delta - F(u_n^\delta)\|_Y^2 \\
&= -\alpha\|y^\delta - F(u_n^\delta)\|_Y^2
\end{aligned}$$

with

$$\alpha := -\lambda\left(\frac{2(\mu+1)}{\tau} - (2-2\mu-\Lambda L^2)\right) > 0.$$

Here we have used the choice of parameters $w_n \in [\lambda, \Lambda]$ and condition (3.4) in the last inequality. The assertion (ii) follows from estimate (3.7). To obtain the first assertion, we first define the set

$$I := \{n \in \mathbb{N} : \|y^\delta - F(u_n^\delta)\|_Y > \tau\delta\}.$$

For any $n \in I$, we see from (3.7) that

$$\|y^\delta - F(u_n^\delta)\|_Y^2 < \frac{1}{\alpha}\left(\|u_n^\delta - \tilde{u}\|_U^2 - \|u_{n+1}^\delta - \tilde{u}\|_U^2\right)$$

and thus

$$\sum_{n\in I} \|y^\delta - F(u_n^\delta)\|_Y^2 < \frac{1}{\alpha}\|u_0 - \tilde{u}\|_U^2 < \infty. \tag{3.8}$$

From the definition of the set I, we get $\|y^\delta - F(u_n^\delta)\|_Y > \tau\delta$ for all $n \in I$ and therefore

$$\sum_{n\in I} \|y^\delta - F(u_n^\delta)\|_Y^2 > \sum_{n\in I}(\tau\delta)^2 = (\tau\delta)^2|I|.$$

This together with (3.8) ensures that the set I is finite. Since $N(\delta) = |I| + 1$, it is also finite. The assertion (i) is valid. □

From now on, we need to differentiate between the noise-free ($\delta = 0$) and the noisy ($\delta > 0$) cases. Thus $u_n^\delta, y_n^\delta := F(u_n^\delta), \ldots$ and $u_n, y_n := F(u_n), \ldots$ are, respectively, generated from algorithm (3.2) corresponding to $\delta > 0$ and $\delta = 0$.

Lemma 3.3. *Let Assumptions (A2) and (A3) be fulfilled. Assume further that λ and Λ are positive constants satisfying*

$$\lambda \le \Lambda, \quad (2 - 2\mu - \Lambda L^2) > 0. \tag{3.9}$$

Then, for any initial guess $u_0 \in B_U(u^\dagger, \rho)$ and the step sizes $w_n \in [\lambda, \Lambda]$, the following estimate holds true

$$\sum_{n=0}^{\infty} \|y^\dagger - F(u_n)\|_Y^2 < \infty. \tag{3.10}$$

Proof. We see from (3.5) with $\tilde{u} := u^\dagger$ and the choice of step sizes w_n that

$$\begin{aligned}\|u_{n+1} - u^\dagger\|_U^2 - \|u_n - u^\dagger\|_U^2 &\le w_n\|y^\dagger - F(u_n)\|_Y \\ &\qquad \left[2\mu\|y^\dagger - F(u_n)\|_Y - (2 - L^2 w_n)\|y^\dagger - F(u_n)\|_Y\right] \\ &\le -\lambda\left(2 - 2\mu - L^2\Lambda\right)\|y^\dagger - F(u_n)\|_Y^2\end{aligned}$$

for all $0 \le n < N(0)$. Consequently, one has

$$\sum_{0 \le n < N(0)} \|y^\dagger - F(u_n)\|_Y^2 \le \frac{1}{\lambda(2 - 2\mu - L^2\Lambda)} \|u_0 - u^\dagger\|_U^2 < \infty. \tag{3.11}$$

If $N(0)$ is infinite, then we get (3.10) from (3.11). Otherwise, (3.10) is also valid since

$$\|y^\dagger - F(u_n)\|_Y = 0$$

for all $n \ge N(0)$. The proof of the lemma is complete. □

We can now obtain a convergence theorem for the noise-free setting. The lines of argumentation in its proof are similar to the ones in [8] as well as in the proof of Lemma 3.7 below with some modifications. The proof is thus omitted.

Theorem 3.4. *Let all of assumptions of Lemma 3.3 hold true. Then algorithm (3.2) corresponding to $\delta = 0$ either stops after finitely many iterations with an iterate coinciding with an element of $S(u^\dagger, \rho)$ or generates a sequence of iterates that converges strongly to an element of $S(u^\dagger, \rho)$ in U.*

3.2 REGULARIZATION PROPERTY

Assertion (ii) in Lemma 3.2 ensures the boundedness of the family $\{u^\delta_{N(\delta)}\}_{\delta>0}$, which together with the reflexivity of U already ensures weak convergence as $\delta \to 0$.

Corollary 3.5. *Assume that all hypotheses of Lemma 3.2 hold and that in addition Assumption (A1) is fulfilled. Then, the sequence $\{u^\delta_{N(\delta)}\}_{\delta>0}$ splits into subsequences, all of which converge weakly to points of $S(u^\dagger, \rho)$ in U as $\delta \to 0^+$. In addition, if $u^\dagger$ is the unique solution of (3.1) in $B_U(u^\dagger, \rho)$, then $\{u^\delta_{N(\delta)}\}_{\delta>0}$ converges weakly to $u^\dagger$ in U.*

Proof. Since $\{u^\delta_{N(\delta)}\}_{\delta>0}$ is bounded in U, there exist a subsequence, also denoted by $\{u^\delta_{N(\delta)}\}_{\delta>0}$, and an element $\bar{u} \in U$ such that

$$u^\delta_{N(\delta)} \rightharpoonup \bar{u} \quad \text{as } \delta \to 0^+.$$

By virtue of Assumption (A1), we have

$$F\left(u^\delta_{N(\delta)}\right) \rightharpoonup F(\bar{u}) \quad \text{as } \delta \to 0^+.$$

Therefore, $y^\delta - F(u^\delta_{N(\delta)}) \rightharpoonup y^\dagger - F(\bar{u})$ in Y. Moreover, from the discrepancy principle, we have

$$\lim_{\delta \to 0} \|y^\delta - F(u^\delta_{N(\delta)})\|_Y = 0,$$

which implies that $F(\bar{u}) = y^\dagger$. We then have $\bar{u} \in S(u^\dagger, \rho)$. From this, the final claim follows by a subsequence–subsequence argument. □

In the remainder of this section, we will show that the Bouligand–Landweber iteration (3.2) is a *strongly* convergent regularization scheme, i.e., the family $\{u^\delta_{N(\delta)}\}_{\delta>0}$ generated by the Bouligand–Landweber iteration (3.2) together with discrepancy principle (3.3) splits into convergent subsequences, all of them converge strongly to elements belonging to $S(u^\dagger,\rho)$. To overcome the discontinuity of the mapping $u \mapsto G_u$, we need the following notion. Let again $\{u^\delta_n\}_{n\in\mathbb{N}}$ be a sequence generated for some $\delta > 0$.

Definition 3.1. An iterative method is said to be *asymptotically stable* if for any $n \geq 0$ and for any sequence $\{\delta_k\}_{k\in\mathbb{N}}$ converging to zero there exists a subsequence $\{\delta_{k'} := \delta_{k'}(n)\}_{k'\in\mathbb{N}}$ of $\{\delta_k\}_{k\in\mathbb{N}}$ such that

$$u_n^{\delta_{k'}} \to \tilde{u}_n \quad \text{as} \quad k' \to \infty$$

and

$$\tilde{u}_n \to \tilde{u} \quad \text{as} \quad n \to \infty$$

for some $\tilde{u} \in S(u^\dagger,\rho)$.

We now show that the Bouligand–Landweber iteration (3.2) is asymptotically stable. The proof consists of a sequence of technical lemmas.

Lemma 3.6. *Assume that Assumptions (A1) to (A4) are verified and that condition (3.4) is valid. Let the initial guess $u_0 \in B_U(u^\dagger,\rho)$ and the step sizes $w_n \in [\lambda,\Lambda]$ be arbitrary. Then, for each fixed $n \geq 0$ and for any sequence $\{\delta_k\}_{k\in\mathbb{N}}$ converging to zero, there exists a subsequence $\{\delta_{k'} := \delta_{k'}(n)\}_{k'\in\mathbb{N}}$ such that*

$$u_n^{\delta_{k'}} \to \tilde{u}_n \quad \text{in } U \text{ as} \quad k' \to \infty, \tag{3.12}$$

where the sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}} \subset B_U(u^\dagger,\rho)$ is defined by

$$\tilde{u}_0 = u_0, \quad \tilde{u}_{n+1} = \tilde{u}_n + w_n(G_{\tilde{u}_n})^*(y^\dagger - F(\tilde{u}_n)) + w_n r_n, \quad n \geq 0, \tag{3.13}$$

for a sequence $\{r_n\}_{n\in\mathbb{N}} \subset Z$ satisfying

(i) $\|r_n\|_U \leq 2L\|y^\dagger - \tilde{y}_n\|_Y$,

(ii) $(r_n, \tilde{u}_n - \tilde{u})_U \leq (-1+\mu)\|y^\dagger - \tilde{y}_n\|_Y^2 - \big(y^\dagger - \tilde{y}_n, G_{\tilde{u}_n}(\tilde{u}_n - \tilde{u})\big)_Y$,

(iii) $|(r_n, \tilde{u}_m - \tilde{u})_U| \leq 2(1+\mu)\|y^\dagger - \tilde{y}_n\|_Y \left[\|y^\dagger - \tilde{y}_n\|_Y + \|\tilde{y}_m - \tilde{y}_n\|_Y\right]$ *for all $m \geq 0$,*

with $\tilde{y}_n := F(\tilde{u}_n)$, any $\tilde{u} \in S(u^\dagger,\rho)$, and constant L given in Assumption (A2).

Proof. We first prove the limit (3.12) by induction on n. Obviously, (3.12) holds for the case $n = 0$. Assume that $\delta_k \to 0$ and $u_n^{\delta_k} \to \tilde{u}_n$ as $k \to \infty$. Let us set

$$a_n^k := (G_{u_n^{\delta_k}})^*(y^{\delta_k} - y_n^k), \qquad a_n := (G_{\tilde{u}_n})^*(y^\dagger - \tilde{y}_n), \qquad \zeta_n^k := a_n^k - a_n$$

with $y_n^k := F(u_n^{\delta_k})$. We have

$$\begin{aligned} \zeta_n^k &= (G_{u_n^{\delta_k}})^*(y^{\delta_k} - y_n^k) - (G_{\tilde{u}_n})^*(y^\dagger - \tilde{y}_n) \\ &= \left[(G_{u_n^{\delta_k}})^*(y^\dagger - \tilde{y}_n) - (G_{\tilde{u}_n})^*(y^\dagger - \tilde{y}_n)\right] + (G_{u_n^{\delta_k}})^*(y^{\delta_k} - y_n^{\delta_k} - y^\dagger + \tilde{y}_n) \\ &= \eta_n^k - \eta_n + b_n^k \end{aligned}$$

with

$$\eta_n^k := (G_{u_n^{\delta_k}})^*(y^\dagger - \tilde{y}_n), \qquad \eta_n := a_n = (G_{\tilde{u}_n})^*(y^\dagger - \tilde{y}_n),$$
$$b_n^k := (G_{u_n^{\delta_k}})^*(y^{\delta_k} - y_n^{\delta_k} - y^\dagger + \tilde{y}_n).$$

Assumption (A1) together with the fact $u_n^{\delta_k} \to \tilde{u}_n$ implies that $y_n^k \to \tilde{y}_n$ as $k \to \infty$. From this and the boundedness of $\{\|G^*_{u_n^{\delta_k}}\|_{\mathcal{L}(Y,U)}\}_{k\in\mathbb{N}}$ for Assumption (A2) one has

$$b_n^k \to 0 \quad \text{in } U \text{ as} \quad k \to \infty. \tag{3.14}$$

From Assumption (A4), we see that $\{\eta_n^k\}_{k\in\mathbb{N}}$ is bounded in Z and so is $\{\eta_n^k - \eta_n\}_{k\in\mathbb{N}}$. Since $Z \hookrightarrow U$ compactly, there exist a function $r_n \in Z$ and a subsequence $\{\delta_{k'}\}_{k'\in\mathbb{N}}$ of $\{\delta_k\}_{k\in\mathbb{N}}$ such that

$$\eta_n^{k'} - \eta_n \to r_n \quad \text{in } U \text{ as} \quad k' \to \infty. \tag{3.15}$$

Moreover, the bound from Assumption (A2) implies for all $k' \geq 0$ that

$$\|\eta_n^{k'} - \eta_n\|_U = \|(G_{u_n^{\delta_{k'}}})^*(y^\dagger - \tilde{y}_n) - (G_{\tilde{u}_n})^*(y^\dagger - \tilde{y}_n)\|_U \leq 2L\|y^\dagger - \tilde{y}_n\|_Y.$$

Combining this with (3.15) yields that

$$\|r_n\|_U = \lim_{k'\to\infty} \|\eta_n^{k'} - \eta_n\|_U \leq 2L\|y^\dagger - \tilde{y}_n\|_Y,$$

which gives assertion (i).

Furthermore, we have

$$\begin{aligned} u_{n+1}^{\delta_{k'}} &= u_n^{\delta_{k'}} + w_n(G_{u_n^{\delta_{k'}}})^* \left(y^{\delta_{k'}} - F(u_n^{\delta_{k'}})\right) \\ &= u_n^{\delta_{k'}} + w_n a_n^{k'} \\ &= u_n^{\delta_{k'}} + w_n a_n + w_n(\eta_n^{k'} - \eta_n) + w_n b_n^{k'}. \end{aligned}$$

Letting $k' \to \infty$ and using the limits (3.14), (3.15), and $u_n^{\delta_{k'}} \to \tilde{u}_n$ implies that

$$u_{n+1}^{\delta_{k'}} \to \tilde{u}_n + w_n a_n + w_n r_n = \tilde{u}_n + w_n(G_{\tilde{u}_n})^*(y^\dagger - \tilde{y}_n) + w_n r_n.$$

By setting $\tilde{u}_{n+1} := \tilde{u}_n + w_n(G_{\tilde{u}_n})^*(y^\dagger - \tilde{y}_n) + w_n r_n$, we obtain the limit (3.12) and relation (3.13). Since u_n^δ belongs to $B_U(u^\dagger, \rho)$, also $\tilde{u}_n \in B_U(u^\dagger, \rho)$.

It remains to prove assertions (ii) and (iii). Let $\tilde{u}$ be arbitrary in $S(u^\dagger, \rho)$. For (ii), we see from (3.15) that

$$\begin{aligned} (r_n, \tilde{u}_n - \tilde{u})_U &= \lim_{k'\to\infty} \left(\eta_n^{k'} - \eta_n, \tilde{u}_n - \tilde{u}\right)_U \\ &= \lim_{k'\to\infty} \left(y^\dagger - \tilde{y}_n, G_{u_n^{\delta_{k'}}}(\tilde{u}_n - \tilde{u})\right)_Y - \left(y^\dagger - \tilde{y}_n, G_{\tilde{u}_n}(\tilde{u}_n - \tilde{u})\right)_Y \\ &= \lim_{k'\to\infty} A_n^{k'} - B_n \end{aligned} \tag{3.16}$$

with

$$A_n^{k'} := \left(y^\dagger - \tilde{y}_n, G_{u_n^{\delta_{k'}}}(\tilde{u}_n - \tilde{u})\right)_Y,$$
$$B_n := \left(y^\dagger - \tilde{y}_n, G_{\tilde{u}_n}(\tilde{u}_n - \tilde{u})\right)_Y.$$

Moreover, it holds that

$$\begin{aligned}
A_n^{k'} &= \left(y^\dagger - y_n^{k'}, G_{u_n^{\delta_{k'}}}(\tilde{u}_n - \tilde{u})\right)_Y + \left(y_n^{k'} - \tilde{y}_n, G_{u_n^{\delta_{k'}}}(\tilde{u}_n - \tilde{u})\right)_Y \\
&= \left(y^\dagger - y_n^{k'}, y^\dagger - y_n^{k'} - G_{u_n^{\delta_{k'}}}(\tilde{u} - \tilde{u}_n)\right)_Y \\
&\quad - \|y^\dagger - y_n^{k'}\|_Y^2 + \left(y_n^{k'} - \tilde{y}_n, G_{u_n^{\delta_{k'}}}(\tilde{u}_n - \tilde{u})\right)_Y \\
&= \left(y^\dagger - y_n^{k'}, y^\dagger - y_n^{k'} - G_{u_n^{\delta_{k'}}}(\tilde{u} - u_n^{\delta_{k'}})\right)_Y - \left(y^\dagger - y_n^{k'}, G_{u_n^{\delta_{k'}}}(u_n^{\delta_{k'}} - \tilde{u}_n)\right)_Y \\
&\quad - \|y^\dagger - y_n^{k'}\|_Y^2 + \left(y_n^{k'} - \tilde{y}_n, G_{u_n^{\delta_{k'}}}(\tilde{u}_n - \tilde{u})\right)_Y \\
&\leq (-1+\mu)\|y^\dagger - y_n^{k'}\|_Y^2 + L\|y^\dagger - y_n^{k'}\|_Y \|u_n^{\delta_{k'}} - \tilde{u}_n\|_U \\
&\quad + L\|y_n^{k'} - \tilde{y}_n\|_Y \|\tilde{u}_n - \tilde{u}\|_U.
\end{aligned}$$

Here where we use Assumptions (A2) and (A3), and the Cauchy–Schwarz inequality to obtain the last estimate. Letting $k' \to \infty$ and using the limits $u_n^{\delta_{k'}} \to \tilde{u}_n$ and $y_n^{k'} \to \tilde{y}_n$ yields that

$$\lim_{k' \to \infty} A_n^{k'} \leq (-1+\mu)\|y^\dagger - \tilde{y}_n\|_Y^2.$$

From this and (3.16), we obtain assertion (ii). For assertion (iii), we get

$$\begin{aligned}
|(r_n, \tilde{u}_m - \tilde{u})_U| &= \lim_{k' \to \infty} \left| \left(\eta_n^{k'} - \eta_n, \tilde{u}_m - \tilde{u}\right)_U \right| \\
&= \lim_{k' \to \infty} \left| \left(y^\dagger - \tilde{y}_n, G_{u_n^{\delta_{k'}}}(\tilde{u}_m - \tilde{u})\right)_Y - \left(y^\dagger - \tilde{y}_n, G_{\tilde{u}_n}(\tilde{u}_m - \tilde{u})\right)_Y \right| \\
&\leq \|y^\dagger - \tilde{y}_n\|_Y \left[\limsup_{k' \to \infty} \|G_{u_n^{\delta_{k'}}}(\tilde{u}_m - \tilde{u})\|_Y + \|G_{\tilde{u}_n}(\tilde{u}_m - \tilde{u})\|_Y \right].
\end{aligned}$$

Due to Assumption (A3), we can apply the (GTCC) to obtain

$$\begin{aligned}
\|G_{u_n^{\delta_{k'}}}(\tilde{u}_m - \tilde{u})\|_Y &\leq \|G_{u_n^{\delta_{k'}}}(u_n^{\delta_{k'}} - \tilde{u})\|_Y + \|G_{u_n^{\delta_{k'}}}(\tilde{u}_m - u_n^{\delta_{k'}})\|_Y \\
&\leq (1+\mu)\|y^\dagger - y_n^{k'}\|_Y + (1+\mu)\|\tilde{y}_m - y_n^{k'}\|_Y \\
&= (1+\mu)\left[\|y^\dagger - y_n^{k'}\|_Y + \|\tilde{y}_m - y_n^{k'}\|_Y\right],
\end{aligned}$$

which implies that

$$\limsup_{k' \to \infty} \|G_{u_n^{\delta_{k'}}}(\tilde{u}_m - \tilde{u})\|_Y \leq (1+\mu)\left[\|y^\dagger - \tilde{y}_n\|_Y + \|\tilde{y}_m - \tilde{y}_n\|_Y\right].$$

Also, we see from (GTCC) that

$$\|G_{\tilde{u}_n}(\tilde{u}_m - \tilde{u})\|_Y \le (1+\mu)\left[\|y^\dagger - \tilde{y}_n\|_Y + \|\tilde{y}_m - \tilde{y}_n\|_Y\right].$$

From the above inequalities, we obtain

$$|(r_n, \tilde{u}_m - \tilde{u})_U| \le 2(1+\mu)\|y^\dagger - \tilde{y}_n\|_Y \left[\|y^\dagger - \tilde{y}_n\|_Y + \|\tilde{y}_m - \tilde{y}_n\|_Y\right],$$

which completes the proof of the lemma. □

Lemma 3.7. *Assume that Assumptions (A1) to (A4) hold. Let us choose positive numbers λ and Λ such that*

$$\lambda \le \Lambda, \quad -1 + \mu + 5\Lambda L^2 < 0. \tag{3.17}$$

Let the initial guess $u_0 \in B_U(u^\dagger, \rho)$ and the step sizes $w_n \in [\lambda, \Lambda]$ be arbitrary. Then the sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ defined by (3.13) converges strongly to an element of $S(u^\dagger, \rho)$ as $n \to \infty$.

Proof. From (3.13), assertions (i) and (ii) of Lemma 3.6 for the case where $\tilde{u} := u^\dagger$, and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \|\tilde{u}_{n+1} - u^\dagger\|_U^2 - \|\tilde{u}_n - u^\dagger\|_U^2 \\ &= 2\left(\tilde{u}_n - u^\dagger, \tilde{u}_{n+1} - \tilde{u}_n\right)_U + \|\tilde{u}_{n+1} - \tilde{u}_n\|_U^2 \\ &= 2w_n\left(G_{\tilde{u}_n}(\tilde{u}_n - u^\dagger), y^\dagger - \tilde{y}_n\right)_Y + 2w_n\left(r_n, \tilde{u}_n - u^\dagger\right)_U + \|\tilde{u}_{n+1} - \tilde{u}_n\|_U^2 \\ &\le 2w_n(-1+\mu)\|y^\dagger - \tilde{y}_n\|_Y^2 + w_n^2\|(G_{\tilde{u}_n})^*(y^\dagger - \tilde{y}_n) + r_n\|_U^2 \\ &\le 2w_n(-1+\mu)\|y^\dagger - \tilde{y}_n\|_Y^2 + 2w_n^2\|(G_{\tilde{u}_n})^*(y^\dagger - \tilde{y}_n)\|_U^2 + 2w_n^2\|r_n\|_U^2 \\ &\le 2w_n(-1+\mu)\|y^\dagger - \tilde{y}_n\|_Y^2 + 2w_n^2L^2\|y^\dagger - \tilde{y}_n\|_Y^2 + 8w_n^2L^2\|y^\dagger - \tilde{y}_n\|_Y^2 \\ &\le 2w_n\|y^\dagger - \tilde{y}_n\|_Y^2\left[-1+\mu+5\Lambda L^2\right] \end{aligned} \tag{3.18}$$

for all $n \ge 0$. Consequently, one has

$$\sum_{n\ge 0}\|y^\dagger - \tilde{y}_n\|_Y^2 \le \frac{1}{2\lambda(1-\mu-5\Lambda L^2)}\|u_0 - u^\dagger\|_U^2 < \infty. \tag{3.19}$$

Inequality (3.18) also yields that $\{\|e_n\|_U\}_{n\in\mathbb{N}}$ for $e_n := u^\dagger - \tilde{u}_n$ is monotonically decreasing and hence

$$\lim_{n\to\infty}\|e_n\|_U = \gamma \tag{3.20}$$

for some $\gamma \ge 0$.

For any $m, l \in \mathbb{N}$ with $m \le l$, we choose k as follows

$$k \in \arg\min_{m\le t\le l}\|y^\dagger - \tilde{y}_t\|_Y. \tag{3.21}$$

The Cauchy–Schwarz inequality gives

$$\|\tilde{u}_m - \tilde{u}_l\|_U^2 \leq 2\left(\|\tilde{u}_m - \tilde{u}_k\|_U^2 + \|\tilde{u}_k - \tilde{u}_l\|_U^2\right). \tag{3.22}$$

Using the identity

$$\|a-b\|_U^2 = \|a-c\|_U^2 - \|b-c\|_U^2 + 2(a-b, c-b)_U$$

yields

$$\begin{aligned}
\|\tilde{u}_m - \tilde{u}_k\|_U^2 &= \|\tilde{u}_m - u^\dagger\|_U^2 - \|\tilde{u}_k - u^\dagger\|_U^2 + 2\left(\tilde{u}_m - \tilde{u}_k, u^\dagger - \tilde{u}_k\right)_U \\
\|\tilde{u}_l - \tilde{u}_k\|_U^2 &= \|\tilde{u}_l - u^\dagger\|_U^2 - \|\tilde{u}_k - u^\dagger\|_U^2 + 2\left(\tilde{u}_l - \tilde{u}_k, u^\dagger - \tilde{u}_k\right)_U.
\end{aligned}$$

Combining this with (3.22) yields that

$$\begin{aligned}
\|\tilde{u}_m - \tilde{u}_l\|_U^2 &\leq 2\left[\|e_m\|_U^2 + \|e_l\|_U^2 - 2\|e_k\|_U^2\right] + 4\left(e_k - e_m, e_k\right)_U + 4\left(e_k - e_l, e_k\right)_U \\
&= a_{m,l,k} + b_{m,l,k}
\end{aligned} \tag{3.23}$$

with

$$a_{m,l,k} := 2\left[\|e_m\|_U^2 + \|e_l\|_U^2 - 2\|e_k\|_U^2\right]$$

and

$$b_{m,l,k} := 4\left(e_k - e_m, e_k\right)_U + 4\left(e_k - e_l, e_k\right)_U.$$

From the limit (3.20), we obtain that

$$a_{m,l,k} \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \tag{3.24}$$

Moreover, we have

$$(e_k - e_m, e_k)_U = \sum_{n=m}^{k-1} (e_{n+1} - e_n, e_k)_U \leq \sum_{n=m}^{k-1} |\left(e_{n+1} - e_n, e_k\right)_U|. \tag{3.25}$$

From (3.13), we obtain $e_{n+1} - e_n = -w_n (G_{\tilde{u}_n})^*(y^\dagger - \tilde{y}_n) - w_n r_n$ and so

$$\begin{aligned}
(e_{n+1} - e_n, e_k)_U &= -w_n\left(y^\dagger - \tilde{y}_n, G_{\tilde{u}_n} e_k\right)_Y - w_n\left(r_n, e_k\right)_U \\
&= w_n\left(y^\dagger - \tilde{y}_n, G_{\tilde{u}_n}(\tilde{u}_k - u^\dagger)\right)_Y + w_n\left(r_n, \tilde{u}_k - u^\dagger\right)_U.
\end{aligned}$$

It follows that

$$|\left(e_{n+1} - e_n, e_k\right)_U| \leq w_n \|y^\dagger - \tilde{y}_n\|_Y \|G_{\tilde{u}_n}(\tilde{u}_k - u^\dagger)\|_Y + w_n \left|\left(r_n, \tilde{u}_k - u^\dagger\right)_U\right|. \tag{3.26}$$

We now estimate the term $\|G_{\tilde{u}_n}(\tilde{u}_k - u^\dagger)\|_Y$. From (GTCC) and the triangle inequality, it follows that

$$\begin{aligned}
\|G_{\tilde{u}_n}(\tilde{u}_k - u^\dagger)\|_Y &\leq \|G_{\tilde{u}_n}(u^\dagger - \tilde{u}_n)\|_Y + \|G_{\tilde{u}_n}(\tilde{u}_k - \tilde{u}_n)\|_Y \\
&\leq \|y^\dagger - \tilde{y}_n\|_Y + \|F(u^\dagger) - F(\tilde{u}_n) - G_{\tilde{u}_n}(u^\dagger - \tilde{u}_n)\|_Y \\
&\qquad + \|G_{\tilde{u}_n}(\tilde{u}_k - \tilde{u}_n)\|_Y \\
&\leq \|y^\dagger - \tilde{y}_n\|_Y + \mu\|y^\dagger - \tilde{y}_n\|_Y + \|G_{\tilde{u}_n}(\tilde{u}_k - \tilde{u}_n)\|_Y \\
&\leq (1+\mu)\|y^\dagger - \tilde{y}_n\|_Y + \|G_{\tilde{u}_n}(\tilde{u}_k - \tilde{u}_n)\|_Y.
\end{aligned} \tag{3.27}$$

Besides, we also see from (GTCC) that

$$\|F(\tilde{u}_k) - F(\tilde{u}_n) - G_{\tilde{u}_n}(\tilde{u}_k - \tilde{u}_n)\|_Y \leq \mu\|F(\tilde{u}_k) - F(\tilde{u}_n)\|_Y$$

and so

$$\begin{aligned}
\|G_{\tilde{u}_n}(\tilde{u}_k - \tilde{u}_n)\|_Y &\leq (1+\mu)\,\|F(\tilde{u}_k) - F(\tilde{u}_n)\|_Y \\
&\leq (1+\mu)\left(\|y^\dagger - F(\tilde{u}_k)\|_Y + \|y^\dagger - F(u_n)\|_Y\right) \\
&\leq 2\,(1+\mu)\,\|y^\dagger - \tilde{y}_n\|_Y.
\end{aligned}$$

This and (3.27) give

$$\|G_{\tilde{u}_n}(\tilde{u}_k - u^\dagger)\|_Y \leq 3(1+\mu)\|y^\dagger - \tilde{y}_n\|_Y. \tag{3.28}$$

On the other hand, from assertion (iii) of Lemma 3.6, we get that

$$\begin{aligned}
\left|\left(r_n, \tilde{u}_k - u^\dagger\right)_U\right| &\leq 2(1+\mu)\|y^\dagger - \tilde{y}_n\|_Y\left[\|y^\dagger - \tilde{y}_n\|_Y + \|\tilde{y}_k - \tilde{y}_n\|_Y\right] \\
&\leq 2(1+\mu)\|y^\dagger - \tilde{y}_n\|_Y\left[\|y^\dagger - \tilde{y}_n\|_Y\right. \\
&\qquad\qquad\qquad \left. + \|\tilde{y}_k - y^\dagger\|_Y + \|\tilde{y}_n - y^\dagger\|_Y\right] \\
&\leq 6(1+\mu)\|y^\dagger - \tilde{y}_n\|_Y^2.
\end{aligned}$$

The combination of this with (3.26) and (3.28) yields that

$$|\,(e_{n+1} - e_n, e_k)_U\,| \leq 9(1+\mu)w_n\|y^\dagger - \tilde{y}_n\|_Y^2,$$

which, together with (3.25), ensures that

$$|\,(e_k - e_m, e_k)_U\,| \leq 9(1+\mu)\sum_{n=m}^{k-1} w_n\|y^\dagger - \tilde{y}_n\|_Y^2 \leq 9(1+\mu)\Lambda\sum_{n=m}^{k-1}\|y^\dagger - \tilde{y}_n\|_Y^2.$$

Similarly, we have

$$|\,(e_k - e_l, e_k)_U\,| \leq 9(1+\mu)\Lambda\sum_{n=k}^{l-1}\|y^\dagger - \tilde{y}_n\|_Y^2.$$

We therefore get

$$b_{m,l,k} = 4\,(e_k - e_m, e_k)_U + 4\,(e_k - e_l, e_k)_U \leq 36(1+\mu)\Lambda\sum_{n=m}^{l-1}\|y^\dagger - \tilde{y}_n\|_Y^2.$$

Combining this with (3.19) yields that

$$\lim_{m\to\infty} b_{m,l,k} = 0. \tag{3.29}$$

The limits (3.24) and (3.29) together with (3.23) imply that $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in U. Thus, there exists a function $\bar{u} \in B_U(u^\dagger, \rho)$ such that $\tilde{u}_n \to \bar{u}$ and hence $F(\tilde{u}_n) \to F(\bar{u})$ by Assumption (A1) as $n \to \infty$. In addition, we see from (3.19) that $y^\dagger - F(\tilde{u}_n) \to 0$ as $n \to \infty$ and hence $y^\dagger = F(\bar{u})$. This implies that $\bar{u} \in S(u^\dagger, \rho)$, which completes the proof. □

The following result is a direct consequence of Lemmas 3.6 and 3.7.

Corollary 3.8. *The Bouligand–Landweber iteration* (3.2) *is asymptotically stable.*

We are now well prepared to prove our main result.

Theorem 3.9 (Regularization property). *Assume that Assumptions (A1) to (A4) hold and $\tau, \lambda, \Lambda > 0$ satisfy conditions* (3.4) *as well as* (3.17). *Let the initial guess $u_0 \in B_U(u^\dagger, \rho)$ and the step sizes $w_n \in [\lambda, \Lambda]$ be arbitrary and let the stopping index $N(\delta)$ be chosen according to the discrepancy principle* (3.3). *Then, the sequence $\{u^\delta_{N(\delta)}\}_{\delta>0}$ splits into convergent subsequences, all of which converge strongly to elements of $S(u^\dagger, \rho)$. Furthermore, if $u^\dagger$ is the unique solution of* (3.1), *then*

$$\lim_{\delta\to 0} \|u^\delta_{N(\delta)} - u^\dagger\|_U = 0.$$

Proof. The proof of the theorem is based on the asymptotic stability of the Bouligand–Landweber iteration (3.2). Let $\{\delta_k\}_{k\in\mathbb{N}}$ be an arbitrary sequence tending to zero as $k \to \infty$. For each pair (δ_k, y^{δ_k}), we denote by $N_k = N(\delta_k)$ the corresponding stopping index determined from the discrepancy principle (3.3).

Assume first that $\{N_k\}_{k\in\mathbb{N}}$ is bounded. Then there exists a subsequence, named also by $\{N_k\}_{k\in\mathbb{N}}$, such that $N_k \to N$ as $k \to \infty$ for some $N \in \mathbb{N}$. Since the set $\{N_k : k \in \mathbb{N}\}$ is finite, we can without loss of generality assume that $N_k = N$ for all $k \in \mathbb{N}$. From the discrepancy principle, one has

$$\|y^{\delta_k} - F(u_N^{\delta_k})\|_Y \le \tau\delta_k. \tag{3.30}$$

By Lemma 3.6, there exists a subsequence, which is also denoted by $\{\delta_k\}_{k\in\mathbb{N}}$, such that

$$u_N^{\delta_k} \to \tilde{u}_N \quad \text{as} \quad k \to \infty.$$

Letting $k \to \infty$ in estimate (3.30) and using the above limit yields that $y^\dagger = F(\tilde{u}_N)$. We thus get $\tilde{u}_N \in S(u^\dagger, \rho)$.

It remains to consider the case where $N_k \to \infty$ as $k \to \infty$. Without loss of generality, we can assume that $\{N_k\}_{k\in\mathbb{N}}$ is monotonically increasing. By virtue of Lemmas 3.6 and 3.7, there exist sequences $\{\delta_k^{(n)}\}_{k\in\mathbb{N}}, n = 0, 1, 2, \dots,$ such that $\{\delta_k^{(0)}\}_{k\in\mathbb{N}} = \{\delta_k\}_{k\in\mathbb{N}}$, $\{\delta_k^{(n+1)}\}_{k\in\mathbb{N}}$ is a subsequence of $\{\delta_k^{(n)}\}_{k\in\mathbb{N}}$, and

$$u_n^{\delta_k^{(n)}} \to \tilde{u}_n \quad \text{as} \quad k \to \infty$$

with $\tilde{u}_n$ satisfying

$$\tilde{u}_n \longrightarrow \tilde{u} \qquad \text{as} \quad n \longrightarrow \infty$$

for some $\tilde{u} \in S(u^\dagger, \rho)$. Therefore, for each $\varepsilon > 0$, there exists an integer n^* such that

$$\|\tilde{u}_{n^*} - \tilde{u}\|_U < \frac{\varepsilon}{2}.$$

It also follows from Lemma 3.6 that a number $\bar{k} \in \mathbb{N}$ exists such that

$$\|u_{n^*}^{\delta_k^{(n^*)}} - \tilde{u}_{n^*}\|_U < \frac{\varepsilon}{2} \quad \text{for all } k \geq \bar{k}. \tag{3.31}$$

Since $N_k \longrightarrow \infty$ as $k \longrightarrow \infty$, there is an integer $\bar{k}_1 \geq \bar{k}$ with $N_k \geq n^*$ whenever $k \geq \bar{k}_1$. From the fact that $\{\delta_k^{(N_k)}\}_{k\in\mathbb{N}, k\geq \bar{k}_1}$ is a subsequence of $\{\delta_k^{(n^*)}\}_{k\in\mathbb{N}}$ and (3.31), it follows that

$$\|u_{n^*}^{\delta_k^{(N_k)}} - u_{n^*}\|_U < \frac{\varepsilon}{2} \quad \text{for all } k \geq \bar{k}_1.$$

Lemma 3.2 implies that

$$\|u_{N_k}^{\delta_k^{(N_k)}} - \tilde{u}\|_U \leq \|u_{n^*}^{\delta_k^{(N_k)}} - \tilde{u}\|_U \leq \|u_{n^*}^{\delta_k^{(N_k)}} - \tilde{u}_{n^*}\|_U + \|\tilde{u}_{n^*} - \tilde{u}\|_U < \varepsilon \quad \text{for all } k \geq \bar{k}_1.$$

We thus have that

$$\lim_{k\to\infty} \|u_{N_k}^{\delta_k^{(N_k)}} - \tilde{u}\|_U = 0.$$

By using a diagonal subsequence arguments, we obtain the desired result. □

4 NUMERICAL EXPERIMENTS

In this section, we present results of numerical experiments illustrating the performance of the Bouligand–Landweber iteration for the model problem (1.1). A short description of our discretization scheme and the solution of the non-smooth PDE (1.1) using a semi-smooth Newton method will be given in the first subsection. The last subsection contains numerical examples.

4.1 DISCRETIZATION AND SEMI-SMOOTH NEWTON METHOD

For the discretization of the non-smooth semilinear elliptic problem (1.1) and the generalized linearization equation (2.8), we shall use standard continuous piecewise linear finite elements (FE), see, e.g., [6, 13] . From now on, we restrict ourselves to the case $\Omega \subset \mathbb{R}^2$. Let us denote by $\mathcal{T}_h$ the triangulation of Ω with the discretization parameter h indicating the fineness of the triangle, i.e., h being the maximum length of the edges of all the triangles of $\mathcal{T}_h$. For each triangulation $\mathcal{T}_h$, let V_h be the finite-dimensional subspace of $H^1_0(\Omega)$ consisting of functions whose restrictions to a triangle $T \in \mathcal{T}$ are polynomials of first degree. By $\{\varphi_j\}_{j=1}^n$ we denote the basis of V_h corresponding to the nodes $P_1, \ldots, P_n$, that is, V_h is spanned by functions $\varphi_1, \ldots, \varphi_n$

and $\varphi_j(P_i) = \delta_{ji}$ with $(\delta_{ji})_{j,i=1}^n$ being the Kronecker delta. By using the mass lumping procedure, we can discretize the non-smooth semilinear elliptic equation (1.1) in weak form as

$$\int_\Omega \nabla y_h \cdot \nabla v_h dx + \frac{1}{3} \sum_{T \in \mathcal{T}_h} |T| \sum_{P_i \in \bar{T}} \max(y_h(P_i), 0) v_h(P_i) = \int_\Omega u_h v_h dx \quad \forall v_h \in V_h, \tag{4.1}$$

where y_h and $u_h \in V_h$ denote the FE-approximation of y and u, respectively. By a slight abuse of notation, we from now on write $y \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$ instead of $(y_h(P_i))_{i=1}^n$ and $(u_h(P_i))_{i=1}^n$, respectively. The discrete equation (4.1) is then equivalent to the nonlinear algebraic system

$$Ay + D \max(y, 0) = Mu \tag{4.2}$$

with the stiffness matrix $A := \left((\nabla\varphi_j, \nabla\varphi_i)_{L^2(\Omega)}\right)_{i,j=1}^n$, the mass matrix $M := \left((\varphi_j, \varphi_i)_{L^2(\Omega)}\right)_{i,j=1}^n$, the lumped mass matrix $D := \frac{1}{3} \operatorname{diag}(\omega_1, \ldots, \omega_n)$, $\omega_i := |\{\varphi_i \neq 0\}|$, and $\max(\cdot, 0) : \mathbb{R}^n \to \mathbb{R}^n$ being the componentwise max-function.

Similarly, the generalized linearization equation (2.8) is discretized as

$$\int_\Omega \nabla \eta_h \cdot \nabla v_h dx + \int_\Omega \mathbb{1}_{\{y>0\}} \eta_h v_h dx = \int_\Omega w_h v_h dx \quad \forall v_h \in V_h. \tag{4.3}$$

Here η_h and w_h stand for the FE-approximation of η and w, respectively. Using the continuity of integrands and the two-dimensional trapezoidal method, the second term in the left hand side in (4.3) can be approximated by

$$\frac{1}{3} \sum_{T \in \mathcal{T}_h} |T| \sum_{P_i \in \bar{T} \cap \{y>0\}} \eta_h(P_i) v_h(P_i) = \frac{1}{3} \sum_{T \in \mathcal{T}_h} |T| \sum_{P_i \in \bar{T}} \mathbb{1}_{\{y(P_i)>0\}} \eta_h(P_i) v_h(P_i)$$

for h small enough. From this and $y(P_i) = y_h(P_i)$, the discrete equation (4.3) can be rewritten as

$$\int_\Omega \nabla \eta_h \cdot \nabla v_h dx + \frac{1}{3} \sum_{T \in \mathcal{T}_h} |T| \sum_{\bar{T} \ni P_i} \mathbb{1}_{\{y_h(P_i)>0\}}(P_i) \eta_h(P_i) v_h(P_i) = \int_\Omega w_h v_h dx \quad \forall v_h \in V_h. \tag{4.4}$$

Again, by a slight abuse of notation, we denote the coefficient vectors $(\eta_h(P_i))_{i=1}^n$ and $(w_h(P_i))_{i=1}^n$ by $\eta \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$, respectively. The discrete equation (4.4) thus becomes the linear algebraic system

$$(A + K_y)\eta = Mw, \tag{4.5}$$

where the matrix K_y is defined by

$$K_y = \frac{1}{3} \operatorname{diag}\left(\omega_i \mathbb{1}_{\{y_i>0\}}\right) \in \mathbb{R}^{n \times n}.$$

Obviously, the variational equations (4.1) and (4.4) as well as the corresponding algebraic systems (4.2) and (4.5) admit unique solutions.

We now show that the non-smooth nonlinear system (4.2) can be approximately solved by a semi-smooth Newton method. Define the mapping $H : \mathbb{R}^n \to \mathbb{R}^n$ by

$$H(y) = Ay + D \max(y, 0).$$

For each $y^{(k)} \in \mathbb{R}^n$, we set

$$M_k := A + DE_k, \qquad E_k := \operatorname{diag}\left(\mathbb{1}_{\{y_i^{(k)} \geq 0\}}\right).$$

Since the component-wise max function is locally Lipschitz and piecewise continuously differentiable in each component, from [28, Props. 2.26, 2.10, 3.5, 3.8] we deduce that M_k is a Newton derivative of H at $y^{(k)}$. We denote the *active set* at $y^{(k)}$ by

$$AC^{(k)} := \left\{i : 1 \leq i \leq n, y_i^{(k)} \geq 0\right\}.$$

We have for any $y \in \mathbb{R}^n$ that

$$y^T M_k y = y^T A y + y^T D E_k y = y^T A y + \sum_{i \in AC^{(k)}} d_{ii} y_i^2 \geq y^T A y \geq c_0 |y|_2^2$$

for some constant $c_0 > 0$ and hence that $\|M_k\|_2 \leq c_0^{-1}$. Here $|y|_2$ and $\|M\|_2$ denote the Euclidean norm of $y \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$, respectively. Due to [28, Prop. 2.12], the semi-smooth Newton iteration

$$M_k \delta y = Mu - H(y^k), \qquad y^{k+1} = y^k + \delta y, \tag{4.6}$$

then converges locally superlinearly to a solution to (4.2).

4.2 NUMERICAL EXAMPLES

We consider $\Omega = [0,1]^2 \subset \mathbb{R}^2$ and use a uniform triangular *Friedrichs–Keller triangulation* with discretization parameter $h = \frac{\sqrt{2}}{2^8}$. The semi-smooth Newton systems are solved by a direct sparse solver, and the semi-smooth Newton iteration for solving non-smooth nonlinear system (4.2) is terminated whenever the active sets corresponding to two consecutive steps coincided with starting points being the origin of $\mathbb{R}^n$.

The exact solution to be reconstructed is defined as

$$\begin{aligned} u^\dagger(x_1,x_2) &= \max(y^\dagger(x_1,x_2), 0) \\ &\quad + \left[4\pi^2 y^\dagger(x_1,x_2) - 2\left((2x_1-1)^2 + 2(x_1-1+\beta)(x_1-\beta)\right)\sin(2\pi x_2)\right]\mathbb{1}_{(\beta,1-\beta]}(x_1) \end{aligned}$$

where

$$y^\dagger(x_1,x_2) = \left[(x_1-\beta)^2(x_1-1+\beta)^2\sin(2\pi x_2)\right]\mathbb{1}_{(\beta,1-\beta]}(x_1)$$

with constant $\beta = 0.005$ is the corresponding exact state; see Figure 1. It is easy to see that $y^\dagger \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies (2.1) for the right-hand side $u^\dagger$ and that $y^\dagger$ vanishes on a set of measure 2β. Therefore, the forward operator F is not Gâteaux differentiable at $u^\dagger$; see Proposition 2.4. We then add random Gaussian noise to $y^\dagger$ to obtain the noisy data y^δ such that

$$\|y^\dagger - y^\delta\|_{L^2(\Omega)} = \delta$$

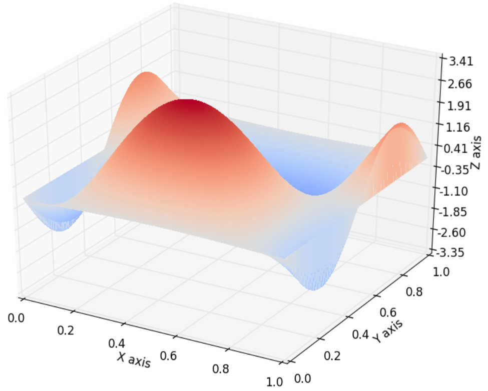


Figure 1: exact data $u^\dagger$

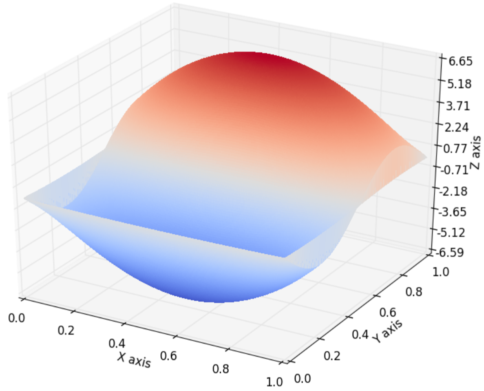


Figure 2: starting guess u_0 given by (4.7)

for a given noise level δ. In the following, we illustrate the convergence for both noise-free and noisy data and two different choices of initial guesses: the trivial guess $u_0 \equiv 0$ and the guess

$$u_0 = u^\dagger - 2\rho \sin(\pi x_1) \sin(2\pi x_2), \tag{4.7}$$

see Figure 2. It is noted that for this starting guess, $u^\dagger$ satisfies the *generalized source condition*

$$u^\dagger - u_0 \in R\left((G_{u^\dagger})^*\right), \tag{4.8}$$

where $R(T)$ denotes the range of operator T. Note also that this choice of u_0 is far from the exact solution $u^\dagger$. In all cases, the parameters in the Bouligand–Landweber iteration (3.2) are set to

$$\mu = 0.1, \quad \tau = 1.4, \quad \rho = 5, \quad w_n = \lambda = \Lambda = \frac{2 - 2\mu}{\bar{L}^2}, \quad \bar{L} = 5 \times 10^{-2}.$$

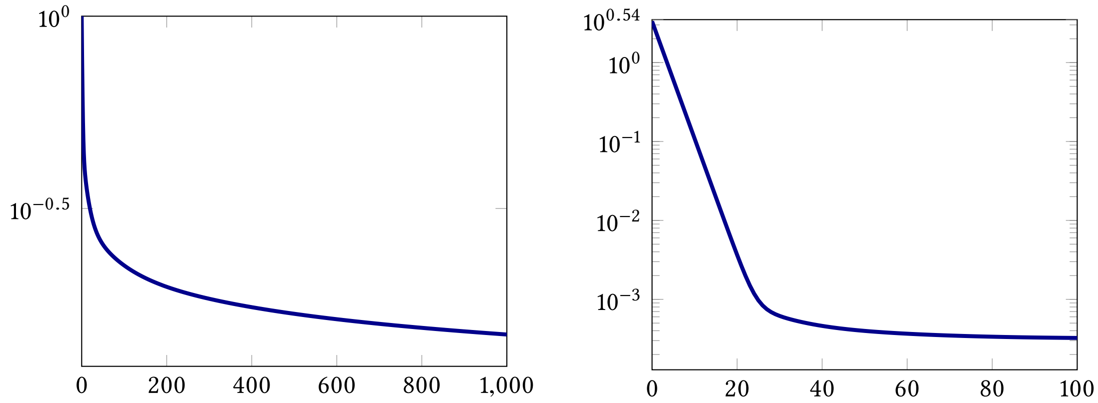
We first address the convergence for noise-free data $y^\dagger$ from Theorem 3.4 by plotting in Figure 3 the relative error

$$\frac{\|u^\dagger - u_n\|_{L^2(\Omega)}}{\|u^\dagger\|_{L^2(\Omega)}} \tag{4.9}$$

of the iterates u_n as a function of the iteration index n. As Figure 3a shows, the iteration slows down for the trivial guess $u_0 \equiv 0$ after 100 steps of rather fast convergence. However, the relative error continues to decrease significantly even after that. In contrast, Figure 3b demonstrates that the rate of convergence for the starting guess (4.7) is substantially greater: Although here the initial relative error is three times greater than for the trivial starting point, the relative error (4.9) drops quickly from 3.33645 to less than 10^{-3} after 20 steps and then continues to reduce.

We next turn to the regularization property from Theorem 3.9. Figure 4 shows the data y^δ and reconstructions $u^\delta_{N(\delta)}$ corresponding to noise level $\delta \in \{10^{-2}, 10^{-3}, 10^{-4}\}$ for the trivial starting guess $u_0 \equiv 0$. The relative error

$$\frac{\|u^\dagger - u^\delta_{N(\delta)}\|_{L^2(\Omega)}}{\|u^\dagger\|_{L^2(\Omega)}} \tag{4.10}$$



(a) starting guess $u_0 \equiv 0$

(b) starting guess u_0 given by (4.7)

Figure 3: relative error (4.9) of iterates in the noise-free setting

of iterates decreases slowly from 0.47704 to 0.26142 to 0.14335 as δ decreases from 10^{-2} to 10^{-3} and 10^{-4}; however, the stopping index $N(\delta)$ is rapidly increasing from 4 to 44 and 1222. This is reasonable as the classical Landweber iteration is known to be similarly slow but reliable.

The noisy data y^δ and reconstructions $u^\delta_{N(\delta)}$ with $\delta \in \{10^{-2}, 10^{-3}, 10^{-5}\}$ for the starting guess (4.7) are shown in Figure 5. Here, as δ reduces from 10^{-2} to 10^{-3} and 10^{-5}, the relative error (4.10) falls rapidly from 0.30378 to 0.02785 and 0.00065, and the stopping index $N(\delta)$ increases only slightly from 7 to 14 and 29. As expected, we thus see much faster convergence for the Bouligand–Landweber iteration if the exact solution satisfies a generalized source condition.

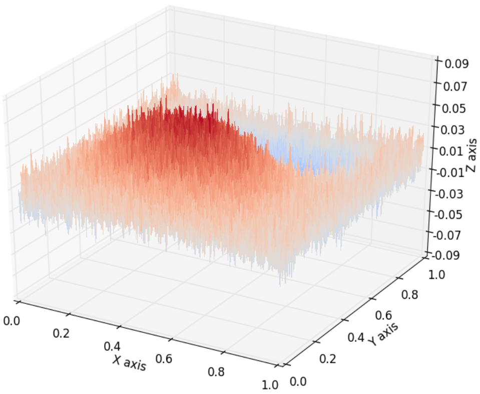
This is shown in more detail in Table 1 for a sequence of noisy data with a noise level δ varying from 10^{-1} to 2×10^{-6}. In particular, the last column shows the empirical convergence rate

$$\frac{\|u^\dagger - u^\delta_{N(\delta)}\|_{L^2(\Omega)}}{\sqrt{\delta}}, \tag{4.11}$$

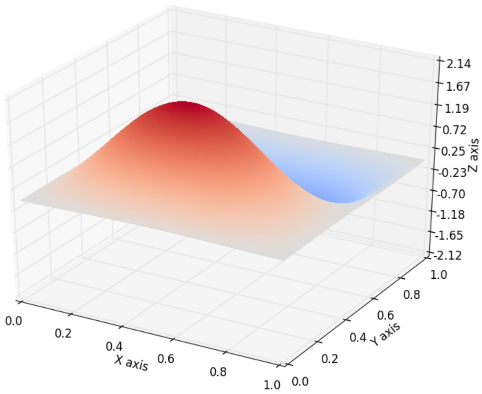
which stabilizes around 0.33 for $\delta < 10^{-4}$. This corresponds to the convergence rate $O(\sqrt{\delta})$ expected from the classical range condition $u^\dagger - u_0 \in R(F'(u^\dagger)^*)$.

5 CONCLUSION

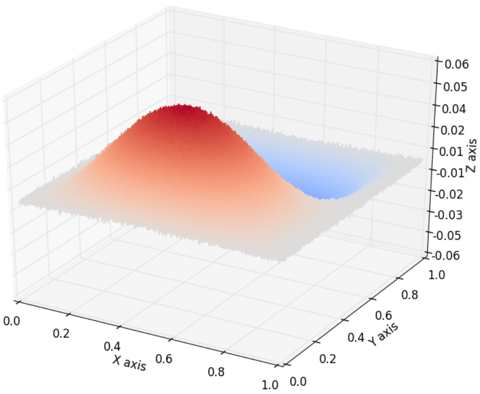
We have considered the iterative regularization of an inverse source problem for a non-smooth elliptic PDE. By considering a Bouligand derivative in place of the non-existent Fréchet derivative of the forward mapping, we derived an implementable regularization method of Landweber-type. If a corresponding generalized tangential cone condition is satisfied – which is the case for our non-smooth model problem provided that the non-differentiability of the forward mapping is sufficiently "weak" at the exact solution – we have shown that the iteration provides a convergent regularization scheme, where we made use of the asymptotic stability of the iteration in place of the missing continuity of the derivative mapping. Numerical examples verify the convergence



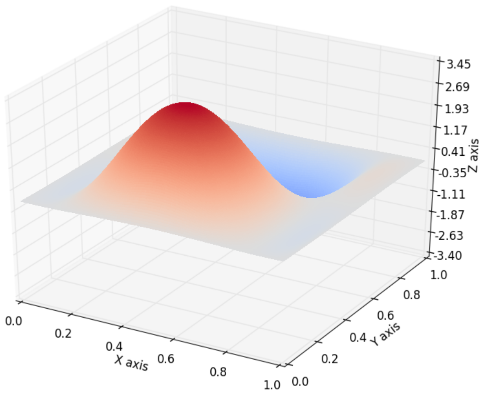
(a) y^δ, $\delta = 10^{-2}$



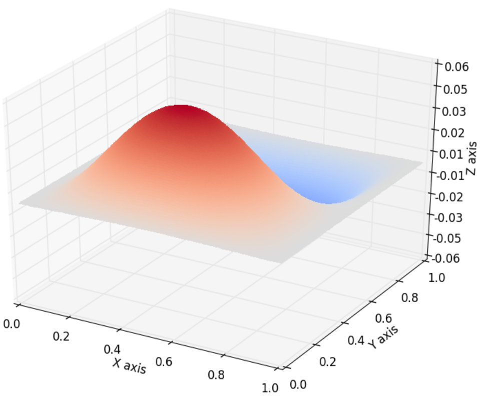
(b) $u^\delta_{N(\delta)}$, $\delta = 10^{-2}$



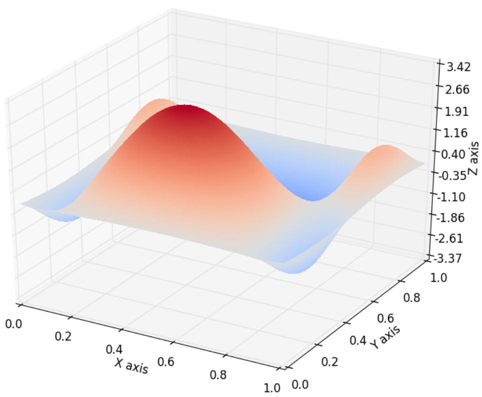
(c) y^δ, $\delta = 10^{-3}$



(d) $u^\delta_{N(\delta)}$, $\delta = 10^{-3}$

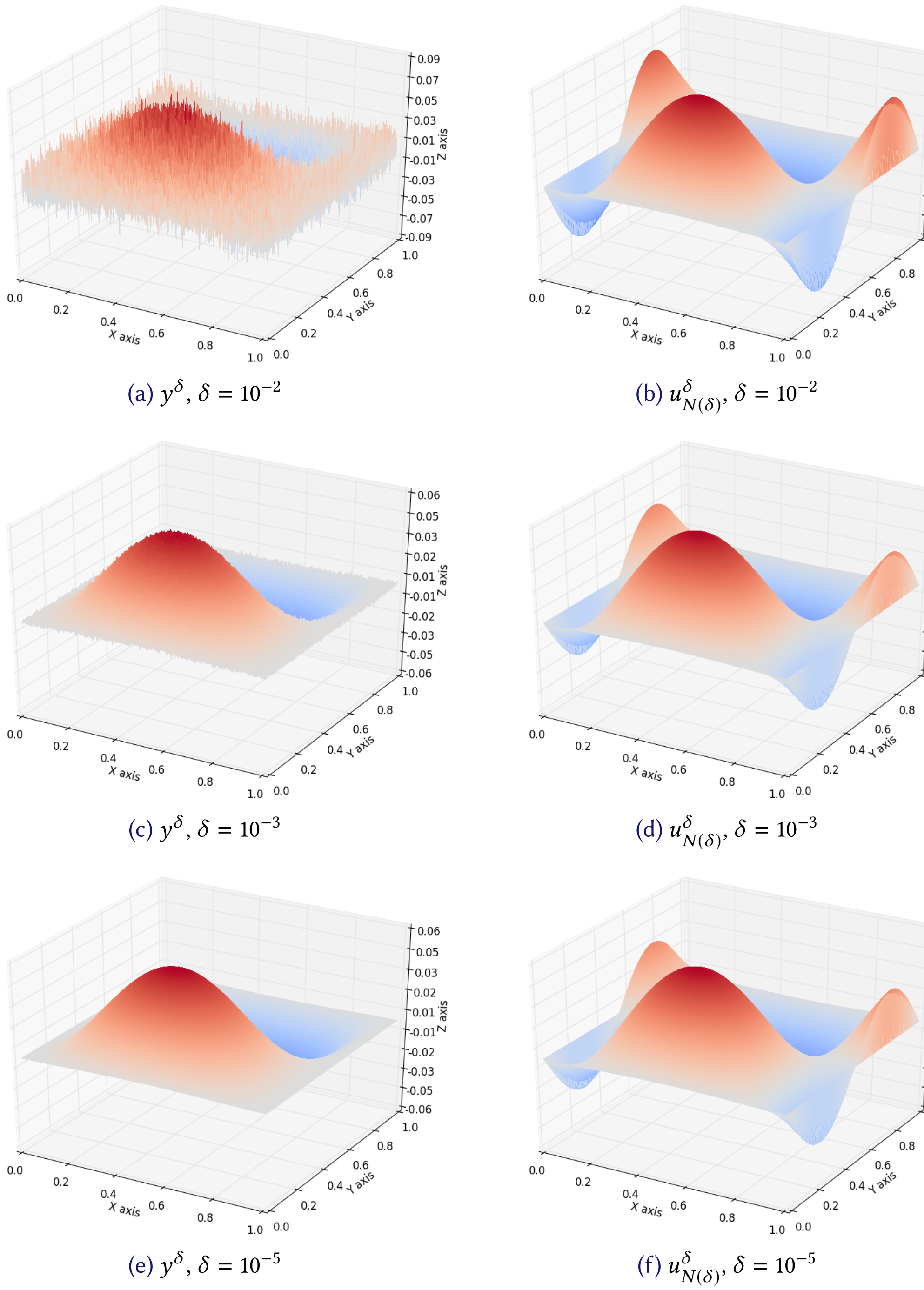


(e) y^δ, $\delta = 10^{-4}$



(f) $u^\delta_{N(\delta)}$, $\delta = 10^{-4}$

Figure 4: noisy data y^δ and reconstructions $u^\delta_{N(\delta)}$ for starting guess $u_0 \equiv 0$



(a) y^δ, $\delta = 10^{-2}$

(b) $u^\delta_{N(\delta)}$, $\delta = 10^{-2}$

(c) y^δ, $\delta = 10^{-3}$

(d) $u^\delta_{N(\delta)}$, $\delta = 10^{-3}$

(e) y^δ, $\delta = 10^{-5}$

(f) $u^\delta_{N(\delta)}$, $\delta = 10^{-5}$

Figure 5: noisy data y^δ and reconstructions $u^\delta_{N(\delta)}$ for starting guess given by (4.7)

Table 1: convergence behavior for starting guess u_0 given by (4.7): noise level δ, stopping index $N(\delta)$, relative error (4.10), empirical convergence rate (4.11)

δ	$N(\delta)$	$\frac{\|u^\delta_{N(\delta)} - u^\dagger\|_{L^2(\Omega)}}{\|u^\dagger\|_{L^2(\Omega)}}$	$\frac{\|u^\delta_{N(\delta)} - u^\dagger\|_{L^2(\Omega)}}{\sqrt{\delta}}$
$1 \cdot 10^{-1}$	1	2.371	$1.124 \cdot 10^{1}$
$5 \cdot 10^{-2}$	3	1.198	8.030
$1 \cdot 10^{-2}$	7	$3.038 \cdot 10^{-1}$	4.552
$5 \cdot 10^{-3}$	9	$1.541 \cdot 10^{-1}$	3.267
$1 \cdot 10^{-3}$	14	$2.785 \cdot 10^{-2}$	1.320
$5 \cdot 10^{-4}$	16	$1.410 \cdot 10^{-2}$	$9.450 \cdot 10^{-1}$
$1 \cdot 10^{-4}$	21	$2.670 \cdot 10^{-3}$	$4.001 \cdot 10^{-1}$
$5 \cdot 10^{-5}$	23	$1.500 \cdot 10^{-3}$	$3.179 \cdot 10^{-1}$
$3 \cdot 10^{-5}$	24	$1.180 \cdot 10^{-3}$	$3.229 \cdot 10^{-1}$
$1 \cdot 10^{-5}$	29	$6.500 \cdot 10^{-4}$	$3.080 \cdot 10^{-1}$
$5 \cdot 10^{-6}$	37	$4.900 \cdot 10^{-4}$	$3.284 \cdot 10^{-1}$
$3 \cdot 10^{-6}$	54	$3.800 \cdot 10^{-4}$	$3.288 \cdot 10^{-1}$
$2 \cdot 10^{-6}$	103	$3.200 \cdot 10^{-4}$	$3.391 \cdot 10^{-1}$

of the iteration for exact as well as for noisy data. While the convergence is slow for an arbitrary initial guess, it is significantly faster for an initial guess for which the exact solution satisfies a generalized source condition.

This work can be extended in a number of directions. First, it would be interesting to derive convergence rates under the generalized source condition (4.8). Another practically relevant issue would be to extend the analysis of the method to cover other classes of non-smooth PDEs such as time-dependent equations or equations with non-smooth nonlinearities entering the higher order terms as for the two-phase Stefan problem. Finally, similar non-smooth extensions of iterative regularization methods of Newton-type should lead to significantly faster convergence.

APPENDIX A ELLIPTIC EQUATIONS WITH PIECEWISE DIFFERENTIABLE NONLINEARITIES

In this appendix, we will show that the generalized tangential cone condition (GTCC) is satisfied for non-smooth semilinear elliptic equations with PC^1-nonlinearities.

We first recall the following definition from, e.g., [24, Chap. 4] and [28, Def. 2.19]. Let $V \subset \mathbb{R}^n$ be an open set. A function $f : V \to \mathbb{R}$ is called a *PC^1-function* (or *piecewise differentiable function*) if f is continuous and for each point $x_0 \in V$ there exist a neighborhood $W \subset V$ and a finite set of C^1-functions $f_i : W \to \mathbb{R}$, $i = 1, 2, \ldots, N$, such that

$$f(x) \in \{f_1(x), f_2(x), \ldots, f_N(x)\} \quad \text{for all} \quad x \in W.$$

The set $\{f_1, f_2, \ldots, f_N\}$ is said to be the *selection functions* of f on W. We denote by $S_f \subset V$ the set of all points in V at which f is not differentiable, i.e.,

$$S_f := \{x \in V : f' \text{ does not exist at } x\}. \tag{A.1}$$

We assume in the following that the set S_f consists of a finite number of points $t_1, t_2, \ldots, t_k$. By virtue of the decomposition theorem for piecewise smooth functions [5, Prop. 2D.7], f can be represented as

$$f(t) = \sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1}, t_i]}(t) f_i(t) \quad \text{for all } t \in \mathbb{R},$$

where f_i, $1 \le i \le k+1$, are C^1-functions on $\mathbb{R}$ and

$$-\infty =: t_0 < t_1 < \cdots < t_k < t_{k+1} := \infty$$

with the convention $(t_k, t_{k+1}] := (t_k, \infty)$. Moreover, we assume that each f_i is non-decreasing on (t_{i-1}, t_i), $1 \le i \le k+1$, and that

$$f_i(t_i) = f_{i+1}(t_i) \quad \text{for all} \quad 1 \le i \le k. \tag{A.2}$$

We require the following technical lemmas regarding the nonlinearity.

Lemma A.1. *For each $M > 0$, let $r_{iM} : [-M, M] \times \mathbb{R} \to [0, \infty)$, $i = 1, 2, \ldots, k+1$, be defined as*

$$r_{iM}(t, s) := \begin{cases} \left| \frac{f_i(t+s) - f_i(t)}{s} - f_i'(t) \right| & s \neq 0, \\ 0 & s = 0, \end{cases} \tag{A.3}$$

Then, r_{iM} is continuous and satisfies

$$r_{iM}(t, s) \to 0 \quad \text{as} \quad s \to 0 \quad \text{uniformly in } t \in [-M, M]. \tag{A.4}$$

Proof. Clearly, r_{iM} is continuous at every points (t, s) with $s \neq 0$. Moreover, we have for any $t \in [-M, M]$ and $s \neq 0$ that

$$r_{iM}(t, s) = \left| \int_0^1 \left(f_i'(t + s\tau) - f_i'(t)\right) d\tau \right| \le \int_0^1 \left| f_i'(t + s\tau) - f_i'(t) \right| d\tau.$$

From this and the uniform continuity of f_i' on bounded sets, Lebesgue's dominated convergence theorem yields the limit (A.4). Consequently, r_{iM} is continuous at $(t, 0)$. □

Let ρ_1 be a positive number. We have the following

Lemma A.2. *We have that*

$$|f_i(t) - f_i(t_i)| \le \beta_i |t - t_i|, \tag{A.5}$$

$$|f_{i+1}(t) - f_{i+1}(t_i)| \le \beta_i |t - t_i| \tag{A.6}$$

for all

$$|t| \le M := \sup \left\{ \|y_u\|_{C(\overline{\Omega})}, |t_i| : u \in B_{L^2(\Omega)}(u^\dagger, \rho_1), 1 \le i \le k \right\}$$

and

$$\beta_i := \max \left\{ \sup\{r_{iM}(t_i, s) : |s| \le 2M\} + |f_i'(t_i)|, \sup\{r_{(i+1)M}(t_i, s) : |s| \le 2M\} + |f_{i+1}'(t_i)| \right\}.$$

Note that the β_i are finite since the functions r_{iM} and $r_{(i+1)M}$ are continuous due to Lemma A.1.

We now consider the non-smooth semilinear elliptic equation

$$\begin{cases} Ay + f(y) = u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{A.7}$$

with $u \in L^2(\Omega)$, A an elliptic second-order partial differential operator with Lipschitz coefficients satisfying

$$c_0\|y\|^2_{H^1_0(\Omega)} \le (Ay, y)_{L^2(\Omega)} \le c_1\|y\|^2_{H^1_0(\Omega)} \quad \text{for all } y \in H^1_0(\Omega)$$

for some $c_1 \ge c_0 > 0$, and a given PC^1-function f satisfying the above assumptions. From [27, Thm. 4.7], we know that for each $u \in L^2(\Omega)$, the equation (A.7) admits a unique solution $y_u \in H^1_0(\Omega) \cap C(\overline{\Omega})$. Furthermore, a constant c_∞ exists such that

$$\|y_u\|_{H^1_0(\Omega)} + \|y_u\|_{C(\overline{\Omega})} \le c_\infty\|u - f(0)\|_{L^2(\Omega)} \quad \forall u \in L^2(\Omega). \tag{A.8}$$

Since $y_u \in C(\overline{\Omega})$, $f(y_u)$ is a function in $C(\overline{\Omega})$ and so $u - f(y_u)$ belongs to $L^2(\Omega)$. Due to [7, Thms. 2.4.2.5 and 3.2.1.2], it follows that $y_u \in H^2(\Omega)$.

From now on, we denote by $F : L^2(\Omega) \to H^2(\Omega) \cap H^1_0(\Omega)$ the solution operator of (A.7). Since the f_i are all C^1-functions, they are thus Lipschitz continuous on bounded sets. From this and [24, Prop. 4.1.2], f is also Lipschitz continuous on bounded sets. By a standard argument, we arrive at the following result, which generalizes the one in Proposition 2.1.

Proposition A.3. *The solution operator $F : L^2(\Omega) \to H^2(\Omega) \cap H^1_0(\Omega)$ is Lipschitz continuous on bounded sets in $L^2(\Omega)$, i.e., for any bounded set $W \subset L^2(\Omega)$ there exists a constant L_W such that*

$$\|F(u) - F(v)\|_{H^2(\Omega)} \le L_W\|u - v\|_{L^2(\Omega)} \quad \forall u, v \in W. \tag{A.9}$$

Now fix $\varepsilon > 0$ such that

$$\varepsilon < \frac{t_i - t_{i-1}}{2} \quad \text{for all } 2 \le i \le k.$$

From the estimates (A.9) and the embedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$, there exists a constant $\bar{\rho} \in (0, \rho_1]$ satisfying

$$\|y_{\hat{u}} - y_u\|_{C(\overline{\Omega})} < \varepsilon \tag{A.10}$$

whenever $u, \hat{u} \in B_{L^2(\Omega)}(u^\dagger, \bar{\rho})$.

For each $u \in L^2(\Omega)$, we further denote by $G_u : L^2(\Omega) \to H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ the solution operator of the linear equation

$$\begin{cases} A\eta + a_u\eta = w & \text{in } \Omega, \\ \eta = 0 & \text{on } \partial\Omega, \end{cases} \tag{A.11}$$

with $w \in L^2(\Omega)$ and

$$a_u(x) := \sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1},t_i]}(y_u(x)) f_i'(y_u(x))$$

for $x \in \Omega$ and $y_u := F(u)$. It is easy to see that

$$a_u(x) \in \partial_B f(y_u(x))$$

for all $x \in \Omega$, where $\partial_B f(t)$ stands for the Bouligand subdifferential of f at t.

Remark A.4. When $f(t) := t^+$, we have $k = 1$, $f_1(t) = 0$, $f_2(t) = t$ and $S_f = \{t_1 := 0\}$. In this case, $a_u = \mathbb{1}_{\{y_u > 0\}}$, and so (for $A = -\Delta$), G_u reduces to the one defined in Proposition 2.5.

From the a priori estimate (A.8), we see that for any bounded set $W \subset L^2(\Omega)$, the set $\{y_u = F(u) : u \in W\}$ is bounded in $C(\overline{\Omega})$. Therefore, there exists a constant C_W satisfying

$$0 \leq a_u(x) \leq C_W$$

for a.e. $x \in \Omega$ and for all $u \in W$. The same lines as in the proof of Lemma 2.6 imply that G_u satisfies the estimates in (2.7) as well as in Lemma 2.6 for all $u \in W$ with some appropriate constants.

We now turn to the verification of the generalized tangential cone condition for F. The following lemma is a generalization of the key Lemma 2.7.

Lemma A.5. *Let $u, \hat{u} \in L^2(\Omega)$ with $u, \hat{u} \in B_{L^2(\Omega)}(u^\dagger, \bar{\rho})$ and $\frac{d}{2} < p < 2$. Then, the following estimate holds*

$$\|F(\hat{u}) - F(u) - G_u(\hat{u} - u)\|_{L^2(\Omega)} \leq L_p |\Omega|^{1/2} \|\zeta(u, \hat{u})\|_{L^{p'}(\Omega)} \|F(\hat{u}) - F(u)\|_{L^2(\Omega)}$$

with $p' = \frac{2p}{2-p}$ for some constant L_p and

$$\zeta(u,\hat{u}) := \sum_{i=1}^{k} \left[\mathbb{1}_{\{y_u \in (t_i - \varepsilon, t_i), y_{\hat{u}} \in (t_i, t_i + \varepsilon)\}} + \mathbb{1}_{\{y_{\hat{u}} \in (t_i - \varepsilon, t_i), y_u \in (t_i, t_i + \varepsilon)\}} \right] \beta_i \\ + \sum_{i=1}^{k} \mathbb{1}_{\{y_u = t_i, y_{\hat{u}} \in (t_i, t_i + \varepsilon)\}} \beta_i + \sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1}, t_i]}(y_u) r_{iM}(y_u, y_{\hat{u}} - y_u).$$

Proof. Let us set $\hat{y} := F(\hat{u})$, $y := F(u)$, $\xi := G_u(\hat{u} - u)$, and $\omega := \hat{y} - y - \xi$. We then have that

$$\begin{aligned} A\hat{y} + f(\hat{y}) &= \hat{u}, \\ Ay + f(y) &= u, \\ A\xi + a_u \xi &= \hat{u} - u. \end{aligned}$$

This implies that

$$A(\hat{y} - y - \xi) + a_u(\hat{y} - y - \xi) = f(y) - f(\hat{y}) + a_u(\hat{y} - y)$$

or equivalently,

$$A\omega + a_u \omega = b$$

with

$$b = f(y) - f(\hat{y}) + a_u(\hat{y} - y).$$

By a computation, we have

$$\begin{aligned} b &= \sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1},t_i]}(y) f_i(y) - \sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1},t_i]}(\hat{y}) f_i(\hat{y}) + \sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1},t_i]}(y) f_i'(y)(\hat{y} - y) \\ &= b_1 + b_2 \end{aligned} \tag{A.12}$$

with

$$b_1 := -\sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1},t_i]}(y) \left(f_i(\hat{y}) - f_i(y) - f_i'(y)(\hat{y} - y) \right)$$

and

$$b_2 := \sum_{i=1}^{k} \left(\mathbb{1}_{(t_{i-1},t_i]}(y) - \mathbb{1}_{(t_{i-1},t_i]}(\hat{y}) \right) f_i(\hat{y}). \tag{A.13}$$

From the definition of r_{iM}, it holds that

$$|b_1| \le \sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1},t_i]}(y) r_{iM}(y, \hat{y} - y)|\hat{y} - y|. \tag{A.14}$$

Because of (A.10), we have

$$\begin{aligned} \mathbb{1}_{(t_{i-1},t_i]}(y) - \mathbb{1}_{(t_{i-1},t_i]}(\hat{y}) = {} & \mathbb{1}_{\{y \in (t_{i-1}, t_{i-1}+\varepsilon), \hat{y} \in (y-\varepsilon, t_{i-1})\}} + \mathbb{1}_{\{y \in (t_i - \varepsilon, t_i), \hat{y} \in (t_i, y+\varepsilon)\}} \\ & - \mathbb{1}_{\{\hat{y} \in (t_{i-1}, t_{i-1}+\varepsilon), y \in (\hat{y}-\varepsilon, t_{i-1})\}} - \mathbb{1}_{\{\hat{y} \in (t_i - \varepsilon, t_i), y \in (t_i, \hat{y}+\varepsilon)\}} \\ & + \mathbb{1}_{\{y \in (t_{i-1}, t_{i-1}+\varepsilon), \hat{y} = t_{i-1}\}} + \mathbb{1}_{\{y = t_i, \hat{y} \in (t_i, y+\varepsilon)\}} \\ & - \mathbb{1}_{\{\hat{y} \in (t_{i-1}, t_{i-1}+\varepsilon), y = t_{i-1}\}} - \mathbb{1}_{\{\hat{y} = t_i, y \in (t_i, \hat{y}+\varepsilon)\}} \end{aligned}$$

with the convention that

$$(-\infty - \varepsilon, -\infty) = (-\infty, -\infty + \varepsilon) = (+\infty - \varepsilon, +\infty) = (+\infty, +\infty + \varepsilon) = \emptyset.$$

This implies

$$\mathbb{1}_{(t_{i-1},t_i]}(y) - \mathbb{1}_{(t_{i-1},t_i]}(\hat{y}) = d_i + e_i \tag{A.15}$$

with

$$\begin{aligned} d_i := {} & \mathbb{1}_{\{y \in (t_{i-1}, t_{i-1}+\varepsilon), \hat{y} \in (y-\varepsilon, t_{i-1})\}} + \mathbb{1}_{\{y \in (t_i - \varepsilon, t_i), \hat{y} \in (t_i, y+\varepsilon)\}} \\ & - \mathbb{1}_{\{\hat{y} \in (t_{i-1}, t_{i-1}+\varepsilon), y \in (\hat{y}-\varepsilon, t_{i-1})\}} - \mathbb{1}_{\{\hat{y} \in (t_i - \varepsilon, t_i), y \in (t_i, \hat{y}+\varepsilon)\}} \\ = {} & \mathbb{1}_{\{\hat{y} \in (t_{i-1}-\varepsilon, t_{i-1}), y \in (t_{i-1}, \hat{y}+\varepsilon)\}} + \mathbb{1}_{\{\hat{y} \in (t_i, t_i+\varepsilon), y \in (\hat{y}-\varepsilon, t_i)\}} \\ & - \mathbb{1}_{\{\hat{y} \in (t_{i-1}, t_{i-1}+\varepsilon), y \in (\hat{y}-\varepsilon, t_{i-1})\}} - \mathbb{1}_{\{\hat{y} \in (t_i - \varepsilon, t_i), y \in (t_i, \hat{y}+\varepsilon)\}} \end{aligned} \tag{A.16}$$

and

$$e_i := \mathbb{1}_{\{y\in(t_{i-1},t_{i-1}+\varepsilon),\hat{y}=t_{i-1}\}} + \mathbb{1}_{\{y=t_i,\hat{y}\in(t_i,t_i+\varepsilon)\}}$$
$$- \mathbb{1}_{\{\hat{y}\in(t_{i-1},t_{i-1}+\varepsilon),y=t_{i-1}\}} - \mathbb{1}_{\{\hat{y}=t_i,y\in(t_i,t_i+\varepsilon)\}}.$$

Multiplying two sides of (A.16) by $f_i(\hat{y})$ and then summing up, we get

$$\sum_{i=1}^{k+1} d_i f_i(\hat{y}) = \sum_{i=1}^{k} \left[\mathbb{1}_{\{\hat{y}\in(t_i,t_i+\varepsilon),y\in(\hat{y}-\varepsilon,t_i)\}} - \mathbb{1}_{\{\hat{y}\in(t_i-\varepsilon,t_i),y\in(t_i,\hat{y}+\varepsilon)\}}\right](f_i(\hat{y}) - f_{i+1}(\hat{y})).$$

Similarly, we obtain

$$\begin{aligned}\sum_{i=1}^{k+1} e_i f_i(\hat{y}) &= \sum_{i=1}^{k} \left[\mathbb{1}_{\{y=t_i,\hat{y}\in(t_i,t_i+\varepsilon)\}} - \mathbb{1}_{\{\hat{y}=t_i,y\in(t_i,t_i+\varepsilon)\}}\right](f_i(\hat{y}) - f_{i+1}(\hat{y}))\\ &= \sum_{i=1}^{k} \mathbb{1}_{\{y=t_i,\hat{y}\in(t_i,t_i+\varepsilon)\}}(f_i(\hat{y}) - f_{i+1}(\hat{y})).\end{aligned}$$

Here we used the identities (A.2) to obtain the last equality. From the above equalities and (A.13) as well as (A.15), it holds that

$$\begin{aligned}|b_2| &\le \sum_{i=1}^{k} \left[\mathbb{1}_{\{\hat{y}\in(t_i,t_i+\varepsilon),y\in(\hat{y}-\varepsilon,t_i)\}} + \mathbb{1}_{\{\hat{y}\in(t_i-\varepsilon,t_i),y\in(t_i,\hat{y}+\varepsilon)\}}\right]|f_i(\hat{y}) - f_{i+1}(\hat{y})|\\ &\quad + \sum_{i=1}^{k} \mathbb{1}_{\{y=t_i,\hat{y}\in(t_i,t_i+\varepsilon)\}}|f_i(\hat{y}) - f_{i+1}(\hat{y})|\\ &\le \sum_{i=1}^{k} \left[\mathbb{1}_{\{\hat{y}\in(t_i,t_i+\varepsilon),y\in(t_i-\varepsilon,t_i]\}} + \mathbb{1}_{\{\hat{y}\in(t_i-\varepsilon,t_i),y\in(t_i,t_i+\varepsilon)\}}\right]|f_i(\hat{y}) - f_{i+1}(\hat{y})|.\end{aligned}$$

Besides, on the set $\{y \in (t_i - \varepsilon, t_i], \hat{y} \in (t_i, t_i + \varepsilon)\}$ we deduce from the non-decreasing monotonicity of f_i and f_{i+1} that

$$f_i(\hat{y}) \ge f_i(t_i) = f_{i+1}(t_i) \le f_{i+1}(\hat{y}),$$

which gives

$$f_{i+1}(t_i) - f_{i+1}(\hat{y}) \le f_i(\hat{y}) - f_{i+1}(\hat{y}) \le f_i(\hat{y}) - f_i(t_i).$$

Consequently,

$$|f_i(\hat{y}) - f_{i+1}(\hat{y})| \le \max\{|f_{i+1}(t_i) - f_{i+1}(\hat{y})|, |f_i(\hat{y}) - f_i(t_i)|\}$$

on the set $\{y \in (t_i - \varepsilon, t_i], \hat{y} \in (t_i, t_i + \varepsilon)\}$. Combining this with (A.5) and (A.6) yields

$$\begin{aligned}\mathbb{1}_{\{y\in(t_i-\varepsilon,t_i],\hat{y}\in(t_i,t_i+\varepsilon)\}}|f_i(\hat{y}) - f_{i+1}(\hat{y})| &\le \mathbb{1}_{\{y\in(t_i-\varepsilon,t_i],\hat{y}\in(t_i,t_i+\varepsilon)\}}\beta_i|\hat{y} - t_i|\\ &\le \mathbb{1}_{\{y\in(t_i-\varepsilon,t_i],\hat{y}\in(t_i,t_i+\varepsilon)\}}\beta_i|\hat{y} - y|.\end{aligned}$$

Similarly, we get

$$\mathbb{1}_{\{\hat{y}\in(t_i-\varepsilon,t_i),y\in(t_i,t_i+\varepsilon)\}}|f_i(\hat{y}) - f_{i+1}(\hat{y})| \le \beta_i \mathbb{1}_{\{\hat{y}\in(t_i-\varepsilon,t_i),y\in(t_i,t_i+\varepsilon)\}}|\hat{y} - y|.$$

These above inequalities give

$$|b_2| \leq \sum_{i=1}^{k} \beta_i \left(\mathbb{1}_{\{y \in (t_i - \varepsilon, t_i], \hat{y} \in (t_i, t_i + \varepsilon)\}} + \mathbb{1}_{\{\hat{y} \in (t_i - \varepsilon, t_i), y \in (t_i, t_i + \varepsilon)\}} \right) |\hat{y} - y|.$$

Combining this with (A.12) and (A.14) yields

$$|b| \leq \zeta(u, \hat{u})|y - \hat{y}|.$$

From this and the same arguments as in Lemma 2.7, we obtain the desired result. □

Using the continuity of F from $L^2(\Omega)$ to $C(\overline{\Omega})$ and the uniform limit (A.4), Lebesgue's dominated convergence theorem implies that the superposition operators $r_{iM} : L^2(\Omega) \to L^{p'}(\Omega)$ defined by (A.3) satisfy

$$r_{iM}(y_u, y_{\hat{u}} - y_u) \to 0 \text{ in } L^{p'}(\Omega) \quad \text{as} \quad u, \hat{u} \to u^\dagger \text{ in } L^2(\Omega)$$

for any $p' \geq 1$. From this and Lemma A.5, we arrive at the following result, whose proof is similar to the one of Corollary 2.8.

Corollary A.6. *Let $\mu > 0$ and assume that $|\{y^\dagger = t_i\}|$ is sufficiently small for all $1 \leq i \leq k$. Then there exists a $\rho > 0$ such that*

$$\|F(\hat{u}) - F(u) - G_u(\hat{u} - u)\|_{L^2(\Omega)} \leq \mu \|F(\hat{u}) - F(u)\|_{L^2(\Omega)}$$

for all $u, \hat{u} \in B_{L^2(\Omega)(u^\dagger, \rho)}$.

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