ERC PROJECT: QUADAG
QUADRATIC REFINEMENTS IN ALGEBRAIC GEOMETRY

This project lies within the mathematical field of *algebraic geometry*, which studies properties of solutions of polynomial equations. As some elementary examples, the equation $3x - 2y = 5$ gives a line, the equation $x^2 + y^2 = 1$ gives a circle, and a general conic (circle, ellipse, parabola, hyperbola) will be the solutions of a general degree two equation, like $3x^2 - 2xy + 7y^2 - 3x + y = 5$. Equations in three variables give surfaces in 3-space and higher dimensional objects are formed by increasing the number of variables. All these are objects of study in algebraic geometry.

My project is concerned with *enumerative geometry*, in which one counts the number of solutions to geometric problems that can be described using polynomial equations. For instance: given four general points and a general line in the plane, how many conics pass through the four points and are tangent to the line?

The answer is two, but only if understood properly. If the four points and the line are defined using real numbers, then there will indeed be two conics that contain the four points and are tangent to the line, but it may be the case that the numbers needed write down the equations of these conics are not real, but are rather complex numbers; you will not ”see” the solutions if you draw a graph in the real plane. If you do not care about this, then the result “two conics” is perfectly satisfactory, but if you do care, then this answer leaves something to be desired.

The main thrust of my project is to refine the simple anwer “two” to a different type of object that is no longer a number, but contains more information. This new object is called a *quadratic form*. A quadratic form is a special kind of degree two polynomial, one that has no degree one or constant terms, for example, the quadratic form $q(x, y, z) = x^2 + 3y^2 - z^2$. A quadratic form has a *rank*, this being essentially the number of variables involved. For a quadratic form with real coefficients, there is also a *signature*, which is essentially the number of positive terms minus the number of negative terms; in our example the rank is three and the signature is one.

For the conics through four points and tangent to a line, there is a way of “counting” the solutions as a quadratic form, where the rank gives the number of solutions allowing complex numbers, while the signature gives the sum of real solutions, each counted with a plus or minus sign, this being a so-called orientation. In this particular example, the quadratic form is $q(x, y) = x^2 - y^2$, with rank two and signature zero, so one has as before two solutions in the complex numbers, and for the real solutions, there are either two, with opposite orientation, or none.

Of course, this is just the very simplest example. Enumerative geometry has been one of the most active fields within algebraic geometry over the past 30 years, and has deep connections with mathematical physics via string theory and mirror symmetry. My hope is that by refining enumerative geometry through the use of quadratic forms, one can shed new light on many mysterious aspects of enumerative geometry over the real numbers and even achieve new results for problems that involve the rational numbers or finite fields.