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Output constrained sliding mode control: A variable gain approach

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Abstract: In this paper output constrained sliding mode control of nonlinear relative degree two systems is considered. The constraints are formulated with respect to the first derivative of the control variable. The system may be effected by matched disturbances. Only the uncertainty bounds are required to be known. A multi-controller approach with variable gains is proposed. The controller guarantees that the constraints can at most be violated for a finite-time, which can be shortened by tuning the controller parameters. Despite the action of the multiple controllers the tracking error of the approach is guaranteed to be bounded.

Keywords: Sliding mode control, Constrained control, Robust control, Nonlinear systems

1. INTRODUCTION

Sliding mode control (SMC) is a well established method for robust control of nonlinear systems. Consider a manifold in the state space. In the reaching phase of SMC the controller drives the states towards the manifold. A sliding mode is said to be established if the states reach the manifold and start to slide on its surface. Using an appropriate definition of the sliding surface, the states can be driven to the origin. Even in the presence of matched disturbances the states remain on the manifold. The main drawback of conventional SMC is occurrence of switching around the manifold denoted as chattering. The chattering may induce unstable dynamics or cause damage. Consequently most of the works in the literature are related to chattering attenuation: There exist approximation techniques like the boundary layer approach (Slotine (1984)) or the exponential reaching law (Gao and Hung (1993), Fallaha et al. (2010)), higher order SMC approaches (Levant (2003), Dinuzzo and Ferrara (2009)), and adaptive gain approaches (Huang et al. (2008), Edwards and Shtessel (2016)). However, only few works about constrained SMC can be found. In Innocenti and Falorni (1998) an admissible region in state space described by a polygonal box is considered. In the region the constraints are satisfied. Sufficient and necessary conditions are established so that the states will remain in the admissible region. An invariant set approach for output constrained SMC is proposed in Richter et al. (2007). First, the unconstrained closed loop dynamics using a conventional SMC are considered. A robust positive invariant (RPI) set, Richter et al. (2007), for this closed loop dynamics is formulated. The RPI set has the form of a cylinder with ellipsoidal cross section. The ellipsoidal cross section can be defined by a positive definite matrix which is found solving a linear matrix inequality (LMI). The LMI is feasible if the closed loop dynamics are asymptotically stable and the switching gain of the controller does not exceed an upper bound. The constrained states are described by a convex set. The system fulfills the output constraints i. e. the states remain in the intersection of the cylinder and the convex set of the constraints, if a number of conditions are fulfilled. The proposed approach is restricted to linear systems, but can consider additive disturbances. An application to slosh control can be found in Richter (2010). A multi-regulator SMC for output constrained systems is proposed in Richter (2011). The controller structure is based on a max-min selection scheme. These schemes are applied in aircraft engine control i. e. to control thrust of turbofan engines under various turbine related limits like temperature or pressure (Litt et al. (2009)). Instead of using linear controllers the max-min scheme proposed in Richter (2011) uses sliding mode controllers. However, the approach is restricted to linear systems. The constraints as well as the control variable are both of realtive degree one. Additionally, the switching gains have to be tuned by experiement. No explicit formulars for the gains are given. In Song et al. (2016) finite time SMC under explicit output constraints is considered. The proposed SMC consists of a proportional, and a switching term. The swichting term ensures that the sliding mode is reached in finite time, whereas the gain of the proportional term can be designed so that the output constraints are fulfilled. However, the corresponding gain has to be obtained from the solution of several LMIs, which are not guaranteed to be feasible. Additionally, the system to be controlled must fulfill the so-called conic sector constraint (ElBsat and Yaz, 2013), which restricts the dynamic behavior of the system and implies that at least nominal system parameters must be known. In Incremona et al. (2016) constraints are expressed as sliding variables using suitable diffeomorphisms. The sliding variables build an integrator chain. Driving the sliding variables to zero will enforce the constraints. Consequently, a higher order SMC is applied to drive the sliding variables to zero. In Rubagotti et al. (2010) SMC and model predictive control (MPC) are combined to incooperate constraints. Integral SMC is applied to compensated matched uncertainties and to facilitate the design of the MPC controller. However, integral SMC requires the input matrix of the system to be precisely known. Numerous works about SMC based

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missile guidance with impact angle constraint like the one of Harl and Balakrishnan (2011) can be found in the literature. Nevertheless, this problem is comparatively easy to solve as satisfying the constraint also guarantees the control goal to be achieved.

In this paper output constraint SMC of nonlinear relative degree two systems is considered. The constraints are formulated with respect to the first derivative of the control variable. A multi-controller approach with variable gains is proposed. The constraint may be violated, similar to a soft-constraint implementation of MPC. However, after finite-time the constraints are guaranteed to be satisfied again. Despite the action of the multiple controllers the tracking error of the approach is guaranteed to be bounded. The paper is organized as follows. In Section 2 the preliminaries and assumptions are discussed. The multi-controller approach is introduced and analyzed in Section 3. A numerical example is considered in Section 4.

2. PROBLEM FORMULATION

Consider a nonlinear system

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})u = f_0(\mathbf{x}) + \phi(\mathbf{x}, t) + g(\mathbf{x})u, \quad (1)$$

$$y_r = h(\mathbf{x}), \quad (2)$$

with states $\mathbf{x} \in \mathbb{R}^n$, control variable $y_r \in \mathbb{R}$, and control input $u \in \mathbb{R}$. Disturbances like parameter uncertainties and variations as well as unknown exogenous inputs are described by ϕ. Based on Lie derivatives the relative degree r_r of the control variable can be defined as

$$r_r \triangleq \min r_r^\star : L_g L_f^{r_r^\star - 1} h(\mathbf{x}) \neq 0. \quad (3)$$

In this paper the relative degree r_r is assumed to be two, and reference tracking

$$\lim_{t\to\infty} y_r(t) = w, \quad (4)$$

subject to constraints being defined by the upper bound

$$y_{c_1} = s_{c_1}\dot{y}_r \leq l_{c_1}, \qquad s_{c_1} = 1, \quad (5)$$

and lower bound

$$y_{c_2} = s_{c_2}\dot{y}_r \leq l_{c_2}, \qquad s_{c_2} = -1, \quad (6)$$

is considered. The reference variable denoted by $w \in \mathbb{R}$ is assumed to be a constant set-point, and the constraints related to the output variables y_{c_i} are defined by constants $0 < l_{c_i} \in \mathbb{R}$, where $c_i \in \{c_1, c_2\}$. The equations (5), (6) imply that only one constraints can be violated at a given time. From (3) it follows that input-output descriptions of the control variable and the output constraints can be achieved as

$$\ddot{y}_r = L_f^2 h(\mathbf{x}) + L_g L_f h(\mathbf{x})u, \quad (7)$$

$$\dot{y}_{c_i} = s_{c_i} L_f^2 h(\mathbf{x}) + s_{c_i} L_g L_f h(\mathbf{x})u. \quad (8)$$

The sliding variable related to the control variable is defined as

$$\sigma_r = -\dot{e}_r - a_0 e_r, \qquad a_0 > 0, \quad (9)$$

where

$$e_r = w - y_r, \quad (10)$$

denotes the tracking error. The sliding variables related to the output constraints are considered to be

$$\sigma_{c_i} = -e_{c_i}, \quad (11)$$

where

$$e_{c_i} = l_{c_i} - y_{c_i}. \quad (12)$$

Based on (9,10,7) the sliding dynamics of σ_r can be obtained as

$$\dot{\sigma}_r = \Psi_r + \Gamma_r u, \quad (13)$$

with

$$\Psi_r = L_f^2 h(\mathbf{x}) + \Omega_r, \quad (14)$$

$$\Omega_r = a_0 \dot{y}_r, \quad (15)$$

$$\Gamma_r = L_g L_f h(\mathbf{x}), \quad (16)$$

where Ψ_r, and Γ_r are assumed to have finite bounds

$$|\Psi_r| \leq \Psi_{r,M}, \qquad 0 < \Gamma_{r_m} \leq \Gamma_r. \quad (17)$$

Parameter variations and external disturbances may appear but can be compensated by the controller as long as the uncertainty bounds hold true. It is assumed that $y_r = h(\mathbf{x})$, and $\dot{y}_r = L_f h(\mathbf{x})$ are continuous functions so that $\sigma_r = \sigma_r(\mathbf{x})$ of (9) is continuous.

3. OUTPUT CONSTRAINED SLIDING MODE CONTROL

In this section the multiple-controller approach is introduced. It is shown that the constraints can only be violated for a finite-time which can be shortened by adjusting the controller parameters. The tracking error is proven to be bounded.

Consider the controller

- If $\forall c_i : y_{c_i} \leq l_{c_i}$ holds true then

  $$u = -k_r \mathrm{sgn}(\sigma_r), \quad (18)$$

  with

  $$k_r = \frac{\eta_r + \Psi_{r,M}\sqrt{2}}{\Gamma_{r,m}\sqrt{2}}, \quad (19)$$

  where $\eta_r > 0$.
- If $\exists \tilde{c}_i : y_{\tilde{c}_i} > l_{\tilde{c}_i}$ holds true then

  $$u = -k_r \mathrm{sgn}(\sigma_r) - s_{\tilde{c}_i} k_{\tilde{c}_i} \mathrm{sgn}(\sigma_{\tilde{c}_i}), \quad (20)$$

  where

  $$k_{\tilde{c}_i} = \begin{cases} \bar{k}_{\tilde{c}_i}, & \text{if } \bar{k}_{\tilde{c}_i} \geq 0, \\ 0, & \text{if } \bar{k}_{\tilde{c}_i} < 0, \end{cases} \quad (21)$$

  with

  $$\bar{k}_{\tilde{c}_i} = \frac{\eta_{\tilde{c}_i} + (\Psi_{r,M} + |\Omega_r|)\sqrt{2}}{\Gamma_{r,m}\sqrt{2}} - s_{\tilde{c}_i} k_r \mathrm{sgn}(\sigma_r)\mathrm{sgn}(\sigma_{\tilde{c}_i}), \quad (22)$$

  and $\eta_{\tilde{c}_i} > 0$, and k_r as defined in (19).

If all the constraints are satisfied the proposed controller reduces to a conventional SMC (18) which drives σ_r to zero. Consequently reference tracking is achieved. If one constraint is violated control law (20) becomes active. Based on the variable gain k_{c_i} it is guaranteed that the constraint can only be violated for a finite-time.

The controller properties are studied as follows.

Proposition 1. For system (1,2) with controller (18-22) sliding variable $\sigma_r(t_2)$ is bounded by

$$|\sigma_r(t_2)| \leq -\frac{\eta_r\sqrt{2}}{2}(t_2 - t_1) + |\sigma_r(t_1)|, \quad \eta_r > 0, \quad (23)$$

if $\forall c_i : y_{c_i}(t^\star) \leq l_{c_i}$, and $\sigma_r(t^\star) \neq 0$, with $t^\star \in [t_1, t_2]$ holds true.

Proof. This is well-known, similar results can be found in e. g. Shtessel et al. (2014). Consider the Lyapunov function candidate $V = 0.5(\sigma_r)^2$. From (13, 18) it follows

$$\begin{aligned}\dot{V} &= \sigma_r(\Psi_r - k_r\Gamma_r\mathrm{sgn}(\sigma_r)) \\ &\leq |\sigma_r|\Psi_{r,M} - k_r\Gamma_{r,m}|\sigma_r|,\end{aligned} \tag{24}$$

As the gain of the proposed controller fulfills

$$k_r \geq \frac{\eta_r + \Psi_{r,M}\sqrt{2}}{\Gamma_{r,m}\sqrt{2}}, \tag{25}$$

it follows

$$\dot{V} \leq -\frac{\eta_r}{\sqrt{2}}|\sigma_r| = -\eta_r V^{1/2}. \tag{26}$$

Integrating inequality (26) from t_1 to t_2 by separating dV, and dt, and inserting $V^{1/2} = \sqrt{0.5}|\sigma_r|$ gives (23). The proposition is proven.

Proposition 2. If $\exists\tilde{c}_i : y_{\tilde{c}_i}(t) > l_{\tilde{c}_i}$ holds true then for system (1,2) with controller (18-22) statement $y_{\tilde{c}_i}(t + t_f) \leq l_{\tilde{c}_i}$ holds true after some finite time $t_f > 0$.

Proof. The statement is proven by contradiction. Assume that no t_f exists and consider the Lyapunov function candidate $V = 0.5(\sigma_{\tilde{c}_i})^2$. Using (8, 11, 12) the sliding dynamics for the constraint are

$$\dot{\sigma}_{\tilde{c}_i} = \Psi_{\tilde{c}_i} + \Gamma_{\tilde{c}_i}\tilde{u}, \tag{27}$$

with

$$\Psi_{\tilde{c}_i} = s_{\tilde{c}_i}L_f^2h(\mathbf{x}), \tag{28}$$

$$\Gamma_{\tilde{c}_i} = L_gL_fh(\mathbf{x}), \tag{29}$$

$$\tilde{u} = s_{\tilde{c}_i}u. \tag{30}$$

From (27, 30, 20) it follows

$$\begin{aligned}\dot{V} &= \sigma_{\tilde{c}_i}(\Psi_{\tilde{c}_i} - s_{\tilde{c}_i}k_r\Gamma_{\tilde{c}_i}\mathrm{sgn}(\sigma_r) - \bar{k}_{\tilde{c}_i}\Gamma_{\tilde{c}_i}\mathrm{sgn}(\sigma_{\tilde{c}_i})), \\ &= (\Psi_{\tilde{c}_i}\mathrm{sgn}(\sigma_{\tilde{c}_i}) - s_{\tilde{c}_i}k_r\Gamma_{\tilde{c}_i}\mathrm{sgn}(\sigma_r)\mathrm{sgn}(\sigma_{\tilde{c}_i}) - \bar{k}_{\tilde{c}_i}\Gamma_{\tilde{c}_i}) \\ &\quad\times|\sigma_{\tilde{c}_i}| \leq -\frac{\eta_{\tilde{c}_i}}{\sqrt{2}}|\sigma_{\tilde{c}_i}| = -\eta_{\tilde{c}_i}V^{1/2},\end{aligned} \tag{31}$$

with constant $\eta_{\tilde{c}_i} > 0$. Then from (31) the inequality

$$\bar{k}_{\tilde{c}_i} \geq \frac{\eta_{\tilde{c}_i}}{\Gamma_{\tilde{c}_i}\sqrt{2}} + \frac{\Psi_{\tilde{c}_i}\mathrm{sgn}(\sigma_{\tilde{c}_i})}{\Gamma_{\tilde{c}_i}} - s_{\tilde{c}_i}k_r\mathrm{sgn}(\sigma_r)\mathrm{sgn}(\sigma_{\tilde{c}_i}), \tag{32}$$

can be obtained. From the triangle inequality

$$|s_{\tilde{c}_i}L_f^2h(\mathbf{x})| - |\Omega_r| \leq |L_f^2h(\mathbf{x}) + \Omega_r| \leq \Psi_{r,M},$$

it follows that $\Psi^{\tilde{c}_i}$ is bounded by

$$|\Psi_{\tilde{c}_i}| \leq \Psi_{r,M} + |\Omega_r|, \tag{33}$$

and $\Gamma_{\tilde{c}_i} = \Gamma_r$ is bounded by

$$0 < \Gamma_{r,m} \leq \Gamma_{\tilde{c}_i}. \tag{34}$$

Based on (33, 34) it follows that inequality (32) holds true if

$$\begin{aligned}\bar{k}_{\tilde{c}_i} \geq & \frac{\eta_{\tilde{c}_i} + (\Psi_{r,M} + |\Omega_r|)\sqrt{2}}{\Gamma_{r,m}\sqrt{2}} \\ & - s_{\tilde{c}_i}k_r\mathrm{sgn}(\sigma_r)\mathrm{sgn}(\sigma_{\tilde{c}_i}),\end{aligned} \tag{35}$$

holds true. Consequently, $\sigma_{\tilde{c}_i}$ goes to zero in finite time. So $y_{\tilde{c}_i}(t + t_f) \leq l_{\tilde{c}_i}$ holds true and the assumption is violated. From (31) it follows that the constraint can only be violated for a finite-time. This finite-time depends on $\eta_{\tilde{c}_i}$, which is a controller parameter that can be tuned. The statement (21) is added to minimize the controller action. The proposition is proven.

Theorem 3. If controller (18-22) is applied to system (1,2) the tracking error e^r is guaranteed to be bounded.

Proof. As (9) is Hurwitz boundedness of the tracking error $e_r(t)$ will be shown by boundedness of the sliding variable $\sigma_r(t)$. From (5, 6, 11, 12) it follows that $\sigma_{c_i}(t) > 0$ holds true if and only if constraint c_i is violated. Consider the sliding variable to be $\sigma_r(t_1)$ at some time t_1. From Proposition 2 it is known that a finite time $t_1^\star \geq t_1$ is guaranteed to exist so that the states are in an admissible region (the constraints are satisfied at time $t_1^\star$).

- Suppose that $\sigma_r(t_1^\star) > 0$:
It will first be shown that as long as $\sigma_r(t_1^\dagger) > 0$ with $t_1^\dagger \in [t_1^\star, t_2]$ for some $t_2 \geq t_1^\star$ holds true an upper bound of $\sigma_r(t_1^\dagger)$ is given by $\sigma_r(t_1^\star)$. At time $t_1^\dagger$ the constraints are satisfied so control input (18) is applied. Considering input (18) with $\mathrm{sgn}(\sigma_r(t_1^\dagger)) = 1$ as $\sigma_r(t_1^\dagger) > 0$, and (27, 28, 29, 30, 14, 16) it follows that the dynamics of the sliding variable related to the constraint c_1 are

$$\begin{aligned}\dot{\sigma}_{c_1}(t_1^\dagger) &= \Psi_r - \Omega_r - \Gamma_rk_r \\ &\leq \Psi_r - \Omega_r - \Gamma_{r,m}k_r.\end{aligned} \tag{36}$$

Consider $\sigma_{c_1}(\bar{t}_1) = 0$ to hold true for some $\bar{t}_1 \in [t_1^\star, t_2]$. It follows from (5, 11, 12) that $\dot{y}_r(\bar{t}_1) = l_{c_1} > 0$. So $\Omega_r(\bar{t}_1) = a_0\dot{y}_r > 0$. For time $\bar{t}_1$ equation (36) can be written as

$$\dot{\sigma}_{c_1}(\bar{t}_1) \leq \Psi_r - \Gamma_{r,m}k_r \leq -\frac{\eta_r}{\sqrt{2}} < 0, \tag{37}$$

where the controller gain from (19) has been considered. Consider $\sigma_{c_1}(t_1^\star) \leq 0$ to hold true as the states are in an admissible region at time $t_1^\star$. It becomes clear that constraint c_1 can never be violated for some $t_1^\dagger \in [t_1^\star, t_2]$, because for constraint c_1 to be violated, $\sigma_{c_1}(t_1^\dagger) > 0$ has to hold. However, this can never be achieved as $\sigma_{c_1}(t_1^\dagger)$ cannot go above zero due to (37). Consequently, one has only to show that the sliding variable $\sigma_r(t_1^\dagger)$ remains bounded if the states are in the admissible region or if constraint c_2 is violated. The former is proven by Proposition 1 and the latter is considered in the following. From (6, 11, 12) it follows

$$\sigma_{c_2}(t_1^\dagger) = -l_{c_2} - \dot{y}_r(t_1^\dagger). \tag{38}$$

As the states are in the admissible region there could exist a time $\bar{t}_2 \in [t_1^\star, t_2]$ with $\sigma_{c_2}(\bar{t}_2) = -l_{c_2} - \dot{y}_r(\bar{t}_2) = 0$. So

$$\dot{y}_r(\bar{t}_2) = -l_{c_2}, \tag{39}$$

holds true. Consider the constraint c_2 to be violated for $\bar{t}_3 \in (\bar{t}_2, t_3)$, where $t_3 \in (\bar{t}_2, t_2]$. Consequently,

$$\dot{y}_r(\bar{t}_3) < -l_{c_2} < 0, \tag{40}$$

can be obtained from (38). Using (9, 10) the sliding variable of the reference is

$$\sigma_r(t) = \dot{y}_r(t) - a_0w + y_r(t). \tag{41}$$

It follows from (39, 40) that the sliding variable for time $\bar{t}_3$ can be expressed as

$$\begin{aligned}\sigma_r(\bar{t}_3) &= \dot{y}_r(\bar{t}_3) - a_0w + y_r(\bar{t}_3), \\ &= \dot{y}_r(\bar{t}_3) - a_0w + y_r(\bar{t}_2) + \int_{\bar{t}_2}^{\bar{t}_3}\dot{y}_r(\tau)d\tau, \\ &\leq \dot{y}_r(\bar{t}_2) - a_0w + y_r(\bar{t}_2), \\ &= \sigma_r(\bar{t}_2) \leq \sigma_r(t_1^\star).\end{aligned} \tag{42}$$

Consider constraint c_2 to be violated in interval $(\bar{t}_2, t_3)$ and to be satisfied for time t_3, i.e. $\dot{y}_r(t_3) = -l_{c_2}$. It follows from (39, 40) that

$$\begin{aligned} \sigma_r(t_3) &= \dot{y}_r(t_3) - a_0 w + y_r(t_3), \\ &= \dot{y}_r(\bar{t}_2) - a_0 w + y_r(\bar{t}_2) + \int_{\bar{t}_2}^{t_3} \dot{y}_r(\tau) d\tau, \\ &\leq \dot{y}_r(\bar{t}_2) - a_0 w + y_r(\bar{t}_2), \\ &= \sigma_r(\bar{t}_2) \leq \sigma_r(t_1^\star), \end{aligned} \tag{43}$$

holds true. From (42), (43) it becomes clear that sliding variable $\sigma_r(t)$ stays bounded from above also if constraint c_2 is violated. So it has been shown that $\sigma_r(t_1^\star)$ is an upper bound for $\sigma_r(t)$ in time interval $[t_1^\star, t_2]$. However, t_2 might be finite if the sliding variable becomes non-positive. Consider $t_1^\diamond$ to be the time at zero-crossing, so that

$$\sigma^r(t_1^\diamond) = 0, \tag{44}$$

holds true. From Proposition 2 it follows that a finite time $t_2^\star \geq t_1^\diamond$ is guaranteed to exist so that the states are in an admissible region. Consider $\sigma_r(t_2^\star)$ to be positive, then an upper bound can be achieved in the same way like it was shown that $\sigma_r(t_1^\star)$ is an upper bound. Consider $\sigma_r(t_2^\star)$ to be negative, then it can be shown that constraint c_2 can never be violated if $\sigma_r(t_2^\dagger) < 0$ and that $\sigma_r(t_2^\dagger)$ remains bounded from below if constraint c_1 is violated. So $\sigma_r(t_2^\dagger)$ is bounded in interval $[t_2^\star, t_5]$. Considering t_5 to be finite there might exist further zero-crossings

$$\sigma_r(t_i^\diamond) = 0, \tag{45}$$

at times $t_i^\diamond$ with $i = 1, ..., n$, which correspond to times $t_{i+1}^\star \geq t_i^\diamond$ at which the states are in the admissible region. Consequently, it can be shown that $\sigma_r(t_{i+1}^\star)$ are upper (lower) bounds like it was shown for $\sigma_r(t_1^\star)$ ($\sigma_r(t_2^\star)$). However, it must be checked that these bounds $|\sigma_r(t_{i+1}^\star)|$ will not go to infinity itself if time increases. This can never happen as based on the zero-crossings

$$\sigma_r(t_{i+1}^\star) = \sigma_r(t_i^\diamond) + \int_{t_i^\diamond}^{t_{i+1}^\star} \dot{\sigma}_r(\tau) d\tau = \int_{t_i^\diamond}^{t_{i+1}^\star} \dot{\sigma}_r(\tau) d\tau,$$

holds true, and $t_{i+1}^\star - t_i^\diamond$ is guaranteed to be finite.

- Suppose that $\sigma_r(t_1^\star) < 0$:
The proof is omitted as it is very similar to the $\sigma_r(t_1^\star) > 0$ case, and uses the same argumentation.
- Suppose that $\sigma_r(t_1^\star) = 0$:
Due to Proposition 2 it is always possible to find times $t_{i+1}^\star = t_i^\star + \Delta t$, with some $\Delta t > \epsilon$, and $i = 1, \ldots, m$ for which the states are in the admissible region. Consider $\sigma_r(t_i^\star) = 0$ to hold true for $t_i^\star$ then $\sigma_r(\bar{t})$ with $\bar{t} \in [t_1^\star, t_i^\star]$ is bounded as $\sigma_r(t)$ is a continuous function. If a time $t_i^\star$ with $\sigma_r(t_i^\star) \neq 0$ exists one may show boundedness like it is shown for the $\sigma_r(t_1^\star) > 0$, $\sigma_r(t_1^\star) < 0$ cases.

The proof of the theorem is completed.

In practice the controller (18-22) is infeasable due to its switching behavior. As stated in Shtessel et al. (2014) a common method to attenuate the chattering effect is to approximate the signum function by a smooth function

$$\mathrm{sgn}(\sigma) \approx \frac{\sigma}{|\sigma| + \epsilon}, \tag{46}$$

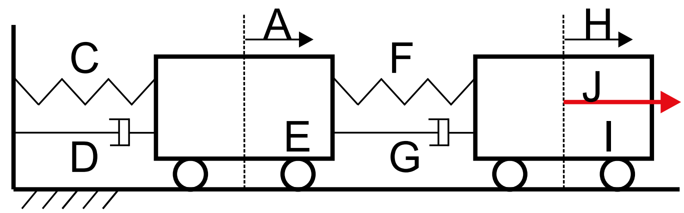

Fig. 1. Two mass spring damper system

with $0 < \epsilon \ll 1$. Consequently, a practical implementation of controller (18-22) is given as

- If $\forall c_i : y_{c_i} \leq l_{c_i}$ holds true then
$$u = -\frac{k_r \sigma_r}{|\sigma_r| + \epsilon_r}, \tag{47}$$
with $k_r = \frac{\eta_r + \Psi_{r,M}\sqrt{2}}{\Gamma_{r,m}\sqrt{2}}$, where $\eta_r > 0$, $\epsilon_r > 0$.
- If $\exists \tilde{c}_i : y_{\tilde{c}_i} > l_{\tilde{c}_i}$ holds true then
$$u = -\frac{k_r \sigma_r}{|\sigma_r| + \epsilon_r} - \frac{s_{\tilde{c}_i} k_{\tilde{c}_i} \sigma_{\tilde{c}_i}}{|\sigma_{\tilde{c}_i}| + \epsilon_{\tilde{c}_i}}, \tag{48}$$
where
$$k_{\tilde{c}_i} = \begin{cases} \bar{k}_{\tilde{c}_i}, & \text{if } \bar{k}_{\tilde{c}_i} \geq 0, \\ 0, & \text{if } \bar{k}_{\tilde{c}_i} < 0, \end{cases} \tag{49}$$
with
$$\bar{k}_{\tilde{c}_i} = \frac{\eta_{\tilde{c}_i} + (\Psi_{r,M} + |\Omega_r|)\sqrt{2}}{\Gamma_{r,m}\sqrt{2}} - \frac{s_{\tilde{c}_i} k_r \sigma_r \sigma_{\tilde{c}_i}}{(|\sigma_r| + \epsilon_r)(|\sigma_{\tilde{c}_i}| + \epsilon_{\tilde{c}_i})}, \tag{50}$$
and $\eta_{\tilde{c}_i} > 0$, $\epsilon_{\tilde{c}_i} > 0$.

Remark 4. Note that the controller

$$u = -k_r \mathrm{sgn}(\sigma_r) - s_{\tilde{c}_i} k_{\tilde{c}_i} \mathrm{sgn}(\sigma_{\tilde{c}_i}), \tag{51}$$

defined in (20) could also be replaced by a controller of the form

$$u = -s_{\tilde{c}_i} k_{\tilde{c}_i}^\star \mathrm{sgn}(\sigma_{\tilde{c}_i}), \tag{52}$$

as $k_{\tilde{c}_i}^\star$ could be chosen appropriate to fulfill the reachability condition (31). However, considering the smooth approximations of (51)

$$u = -k_r \frac{\sigma_r}{|\sigma_r| + \epsilon_r} - s_{\tilde{c}_i} k_{\tilde{c}_i} \frac{\sigma_{\tilde{c}_i}}{|\sigma_{\tilde{c}_i}| + \epsilon_{c_i}}, \tag{53}$$

and (52)

$$u = -s_{\tilde{c}_i} k_{\tilde{c}_i}^\star \frac{\sigma_{\tilde{c}_i}}{|\sigma_{\tilde{c}_i}| + \epsilon_{c_i}}, \tag{54}$$

it can be seen that for $\sigma_{\tilde{c}_i} \to 0$ the transition of (47) to (48) is smooth if (48) equals (53) whereas it is not smooth if (48) equals (54).

4. EXAMPLE

In this section the proposed controller is applied to a nonlinear mechanical mass spring damper system (Fig. 1). Based on Newtons second law the dynamics of the system can be derived as

Table 1. System parameters

Parameter	Symbol	Value
Weight (first mass)	m_1	10 kg
Weight (second mass)	m_2	5 kg
Linear stiffness (first spring)	k_1	20 kg/s^2
Nonl. stiffness (first spring)	k_2	10 kg/(s^{2}m^2)
Linear stiffness (second spring)	k_3	15 kg/s^2
Nonl. stiffness (second spring)	k_4	10 kg/(s^{2}m^2)
Nonl. damping (first damper)	d_1	10 kg/m
Nonl. damping (second damper)	d_2	2 kg/m

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} x_2 \\ \dfrac{F_1(\mathbf{x}) + F_2(\mathbf{x})}{m_1} \\ x_4 \\ -\dfrac{F_2(\mathbf{x})}{m_2} \end{bmatrix}}_{f(\mathbf{x})} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{m_2} \end{bmatrix}}_{g} u, \tag{55}$$

$$F_1(\mathbf{x}) = -k_1 x_1 - k_2 x_1^3 - d_1 x_2^2,$$
$$F_2(\mathbf{x}) = k_3(x_3 - x_1) + k_4(x_3 - x_1)^3 + d_2(x_4 - x_2)^2,$$

where x_1 is the position of the first mass, x_2 is the velocity of the first mass, x_3 is the position of the second mass, and x_4 is the velocity of the second mass. The control input is the force $u = F_u$. The control variable is

$$y_r = h(\mathbf{x}) = x_3. \tag{56}$$

The parameters of the system are given in Table 1. From (55, 56) the input-output description

$$\ddot{y}_r = L_f^2 h(\mathbf{x}) + L_g L_f h(\mathbf{x}) F_u \tag{57}$$

of the control variable can be achieved, where

$$L_f^2 h(\mathbf{x}) = -\frac{F_2(\mathbf{x})}{m_2}, \qquad L_g L_f h(\mathbf{x}) = \frac{1}{m_2}, \tag{58}$$

and the relative degree is two ($r_r = 2$). Following (9) the sliding variable is defined as

$$\sigma_r = -\dot{e}_r - a_0 e_r. \tag{59}$$

From (59, 57) the dynamics of the sliding variable are obtained as

$$\dot{\sigma}_r = \Psi_r + \Gamma_r F_u, \tag{60}$$

with

$$\Psi_r = -\frac{F_2(\mathbf{x})}{m_2} + a_0 x_4, \tag{61}$$

$$\Omega_r = a_0 x_4, \qquad \Gamma_r = \frac{1}{m_2}. \tag{62}$$

The bounds of Ψ_r, and Γ_r are assumed to be

$$\Psi_{r,M} = 0.8, \qquad \Gamma_{r,m} = 0.002. \tag{63}$$

Following (5, 6) the constraints and their corresponding bounds are defined as

$$y_{c_1} = \dot{y}_r = x_4, \qquad s_{c_1} = 1, \tag{64}$$
$$y_{c_2} = -\dot{y}_r = -x_4, \qquad s_{c_2} = -1, \tag{65}$$
$$y_{c_1} \leq 0.4\,\text{m/s} = l_{c_1}, \tag{66}$$
$$y_{c_2} \leq 0.2\,\text{m/s} = l_{c_2}, \tag{67}$$

and based on (11) the sliding variables are defined as

$$\sigma_{c_1} = -0.4 + x_4, \tag{68}$$
$$\sigma_{c_2} = -0.2 - x_4. \tag{69}$$

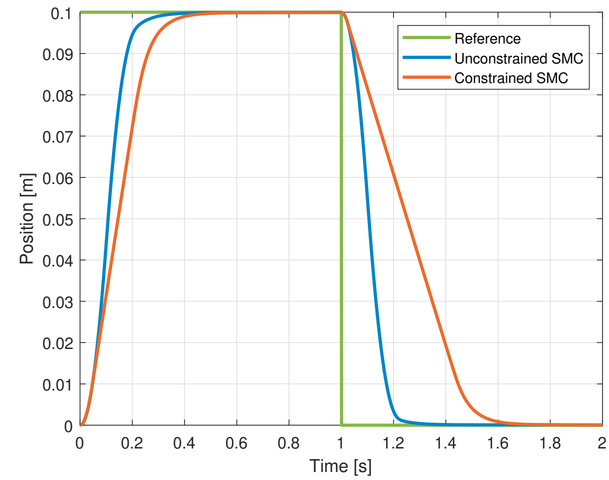

Fig. 2. Visualization of reference tracking

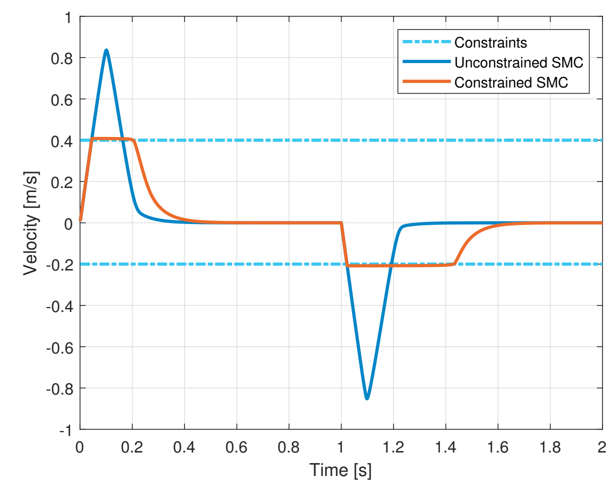

Fig. 3. Consideration of output constraints

The nonlinear system (55) is simulated based on Euler method using sampling time $T_s = 0.001\,\text{s}$ and initial state

$$\mathbf{x}(t_0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T. \tag{70}$$

A simulation duration of 2 s is considered and the reference values for tracking are

$$w = \begin{cases} 0.1\,\text{m} & \text{if } t \leq 1\,\text{s}, \\ 0\,\text{m} & \text{if } 1\,\text{s} < t \leq 2\,\text{s}. \end{cases} \tag{71}$$

The proposed output constrained sliding mode controller (constrained SMC) defined by (47-50) is compared with a standard unconstrained SMC

$$u = -\frac{k_r \sigma_r}{|\sigma_r| + \epsilon_r}, \tag{72}$$

where $k_r = \frac{\eta_r + \Psi_{r,M}\sqrt{2}}{\Gamma_{r,m}\sqrt{2}}$, $\eta_r > 0$, $\epsilon_r > 0$. The controller parameters used in the simulation are: $\eta_r = \eta_{c_1} = \eta_{c_2} = 0.2$, $\epsilon_r = 0.05$, $\epsilon_{c_1} = 0.06$, $\epsilon_{c2} = 0.03$. For the sliding dynamics a_0 is chosen as $a_0 = 15$.

The simulation results are visualized in Figures (2-5). The proposed constrained SMC approach yields only slight violation of the output constraints (similar to a soft constraint

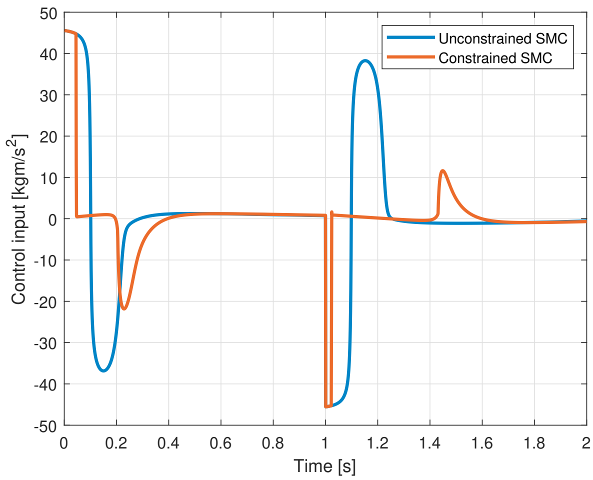

Fig. 4. Smoothness of control inputs

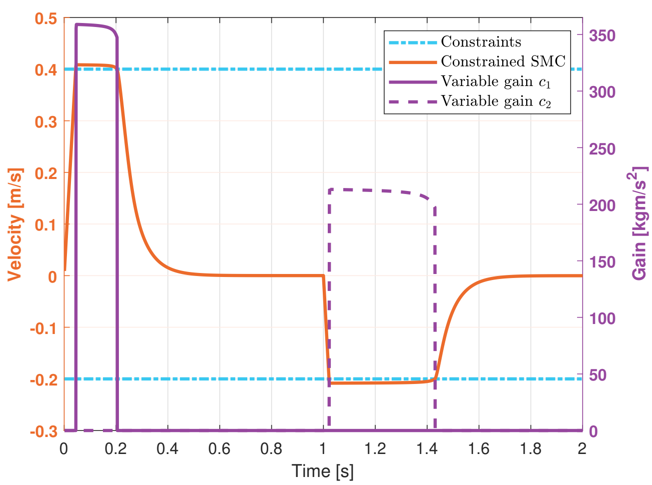

Fig. 5. Visualization of variable gains

implementation of MPC), whereas the conventional SMC shows unconstrained behavior (Fig. 3). Despite the multi-controller action constrained SMC approach achieves reference tracking (Fig. 2). The control input of the unconstrained SMC shows more smooth behavior in comparison to the proposed constrained SMC approach (Fig. 4). This is due to the point that the second term in (48) induces a switching each time a constraint is newly violated. The variable gains of constrained SMC approach are visualized in Fig. 5.

5. CONCLUSION

A multi-controller approach with variable gains has been developed. The controller guarantees the output constraints to be violated at most for a finite-time which can be shortened by tuning of the controller parameters. Despite the multiple controller action the tracking error is bounded and reference tracking is achieved in simulation. As a next step the control approach is intended to be improved so that violation of the constraints can be avoided at all and minimization of the tracking error can be achieved.

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