On t-designs and s-resolvable t-designs from hyperovals

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Abstract

Hyperovals in projective planes turn out to have a link with t-designs. Motivated by an unpublished work of Lonz and Vanstone, we present a construction for t-designs and s-resolvable t-designs from hyperovals in projective planes of order 2^n. We prove that the construction works for $t \leq 5$. In particular, for $t = 5$ the construction yields a family of 5-$(2^n + 2, 8, 70(2^{n-2} - 1))$ designs. For $t = 4$ numerous infinite families of 4-designs on $2^n + 2$ points with block size $2k$ can be constructed for any $k \geq 4$. The construction assumes the existence of a 4-$(2^{n-1} + 1, k, \lambda)$ design, called the indexing design, including the complete 4-$(2^{n-1} + 1, k, \binom{2^{n-1}-3}{k-4})$ design. Moreover, we prove that if the indexing design is s-resolvable, then so is the constructed design. As a result, many of the constructed designs are s-resolvable for $s = 2, 3$. We include a short discussion on the simplicity or non-simplicity of the designs from hyperovals.

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1 Introduction

An *oval* in a projective plane of order q is a set of $q + 1$ points, no three of which are collinear. A *hyperoval* $\mathcal{O}$ in a projective plane of even order $q = 2^n$ is a set of $q + 2$ points such that every line of the plane intersects $\mathcal{O}$ in either zero or in two points. It is a well-known fact that if a finite projective plane contains a hyperoval, then its order must be even. In addition, an oval in a projective plane of even order can uniquely be extended to a hyperoval. The union of a conic and its nucleus is an example of a (regular) hyperoval in the Desarguesian projective plane, $\mathrm{PG}(2, q)$, q even.

Lonz and Vanstone showed in an unpublished work that hyperovals can be used to construct 5-$(2^n + 2, 6, 15)$ designs for every integer $n \geq 3$, see [8]. Inspired by this work we investigate the construction of t-designs by hyperovals in the current paper. We show that t-designs with $t \leq 5$ can be constructed by hyperovals. More precisely, if for any given $k \geq 4$ there exists a 4-$(2^{n-1}+1, k, \lambda)$ design, then there exists a 4-$(2^n+2, 2k, \Lambda)$ design, where the index Λ depends on n, k, λ. For $t=5$, it can be shown that there exists a 5-$(2^n+2, 8, 70(2^{n-2}-1))$ design for $n \geq 4$.

The construction by hyperovals essentially assumes the existence of a certain design, called the indexing design. In particular, if the indexing t-design is s-resolvable, then the constructed design is also s-resolvable. The result is interesting regarding the question of finding s-resolvable t-designs, about which not much is known, see [10, 11]. Actually we have obtained many s-resolvable infinite families of 4- and 5-designs, for $s = 2, 3$ with the construction.

The question of simplicity or non-simplicity of the constructed designs by hyperovals remains an open problem, which appears to be very involved. We will discuss the problem at the end of the paper. However, in the case of the Lonz-Vanstone 5-$(2^n+2, 6, 15)$ design, Jungnickel and Vanstone [6] have proved that when using a regular hyperoval in $\mathrm{PG}(2, 2^n)$ this design is neither simple nor a multiple of a 5-design with a smaller index for $n \geq 4$.

We assume that the reader is familiar with the concepts of t-designs and projective planes. For completeness we include the following definition, see also [10, 11].

Definition 1.1 *A t-(v,k,λ)-design $(X, \mathcal{B})$ is said to be s-resolvable, or to have an s-resolution, with $0 < s < t$, if its block set $\mathcal{B}$ can be partitioned into $N \geq 2$ classes $\mathcal{A}_1, \ldots, \mathcal{A}_N$ such that each $(X, \mathcal{A}_i)$ is an s-(v,k,δ) design for $i = 1, \ldots, N$. Each $\mathcal{A}_i$ is called an s-resolution class or simply a resolution class. The set of N classes is called an s-resolution of the design.*

In particular, note that the s-resolvability of the complete k-$(v,k,1)$ design coincides with the concept of *large set* of s-designs. For more information about s-resolvable t-designs, see [11] for example.

1.1 Description of the construction

We begin by describing our construction for t-designs from hyperovals.

Let $(\mathcal{P}, \mathcal{L})$ be a projective plane of order $q = 2^n$, with point set $\mathcal{P}$ and line set $\mathcal{L}$. Let $\mathcal{O}$ be a hyperoval of $(\mathcal{P}, \mathcal{L})$ consisting of 2^n+2 points of $\mathcal{P}$. Let $(\mathcal{X}, \mathcal{B})$ be a t-$(2^{n-1}+1, k, \lambda)$ design, called *the indexing design.*

- For $x \in \mathcal{P} \setminus \mathcal{O}$ define

$$\begin{aligned} \mathcal{L}_x &= \{L \in \mathcal{L} \ / \ x \in L \text{ and } |L \cap \mathcal{O}| = 2\} \\ \Pi_x &= \{L \cap \mathcal{O} \ / \ L \in \mathcal{L}_x.\} \end{aligned}$$

  Then $|\mathcal{L}_x| = 2^{n-1}+1$ and Π_x is a partition of $\mathcal{O}$ into $(2^{n-1}+1)$ 2-subsets.

- Conceptually, we consider the $(2^{n-1}+1)$ 2-subsets of Π_x as **points**, say $X_1, \ldots, X_{2^{n-1}+1}$. Let $(\Pi_x, \mathcal{D}_x)$ be a copy of the indexing design $(\mathcal{X}, \mathcal{B})$ defined on Π_x and define

$$\mathcal{D} = \bigcup_{x \in \mathcal{P} \setminus \mathcal{O}} \mathcal{D}_x.$$

We will prove that for a suitable indexing design $(\mathcal{X}, \mathcal{B})$, the pair $(\mathcal{O}, \mathcal{D})$ will form a t-$(2^n + 2, 2k, \Lambda)$ design, in which each **point** X_i is replaced by its corresponding two points of the hyperoval.

The Lonz-Vanstone 5-$(2^n + 2, 6, 15)$ design in this context corresponds to the case, where the indexing design $(\mathcal{X}, \mathcal{B})$ is the trivial 3-$(2^{n-1} + 1, 3, 1)$ design defined on Π_x. Thus the blocks of $(\Pi_x, \mathcal{D}_x)$ consist of the collection of all 3-subsets of Π_x. Each 3-subset $D = \{X_i, X_j, X_h\} \subseteq \Pi_x$ with $i \neq j \neq h \neq i$ is a block of $\mathcal{D}_x$. Expanding X_i, X_j, X_h to their corresponding 2-subsets of the hyperoval will give $D = \{a, b, c, d, e, f\}$, where $X_i = \{a, b\}$, $X_j = \{c, d\}$, $X_h = \{e, f\}$. And the three lines $\{a, b\}$, $\{c, d\}$, $\{e, f\}$ are concurrent at the point $x \in \mathcal{P} \setminus \mathcal{O}$.

It is not difficult to see that $(\mathcal{O}, \mathcal{D})$ is a 5-$(2^n + 2, 6, 15)$ design. This can be seen as follows. Let $a, b, c, d, e \in \mathcal{O}$ be any five distinct points of $\mathcal{O}$. There are 15 pairs of lines (L_1, L_2) that can be formed in the projective plane from these 5 points, having their intersection points not in $\mathcal{O}$. Let assume $L_1 = \{a, b\}$, $L_2 = \{c, d\}$. Then the lines L_1 and L_2 will meet at a point $x \in \mathcal{P} \setminus \mathcal{O}$. The line $\{x, e\}$ will meet $\mathcal{O}$ at another point f. And $\{a, b, c, d, e, f\}$ is a block of $\mathcal{D}$. Hence a, b, c, d, e appear together in 15 blocks of $\mathcal{D}$, as desired.

2 A family of $\mathbf{5 - (2^n + 2, 8, 70(2^{n-2} - 1))}$ designs

By observing the construction of 5-$(2^n + 2, 6, 15)$ above, we notice that if the complete 4-$(2^{n-1} + 1, 4, 1)$ design is used as the indexing design, then the construction will work and yields 5-designs with block size 8. We record the result in the following theorem.

Theorem 2.1 *There exists a* 5-$(2^n + 2, 8, 70(2^{n-2} - 1))$ *design for every* $n \geq 4$.

Proof. For any given point $x \in \mathcal{P} \setminus \mathcal{O}$, choose the complete $4 - (2^{n-1} + 1, 4, 1)$ design as the indexing design defined on Π_x. Note that for this indexing design $(\Pi_x, \mathcal{D}_x)$ we have $\lambda_4 = 1$, $\lambda_3 = (2^{n-1} - 2)$.

Let $a, b, c, d, e \in \mathcal{O}$ be any 5 points. There are two cases to be considered.

- Fixing a pair of lines, say, $\{a, b\}$ and $\{c, d\}$. There are 15 possible choices for such pairs from the 5 points a, b, c, d, e. Now lines $\{a, b\}$ and $\{c, d\}$ intersect at a unique point $x \in \mathcal{P} \setminus \mathcal{O}$. This shows that in this case a, b, c, d, e appear together $15.\lambda_3 = 15.(2^{n-1} - 2)$ times in the blocks of $\mathcal{D}$.

- Fixing a line, say $\{a, b\}$. There are 10 possible choices of such lines from a, b, c, d, e. There are $(2^n - 1)$ points on line $\{a, b\}$ outside $\mathcal{O}$. The three intersection points of line $\{a, b\}$ with lines $\{c, d\}$, $\{c, e\}$ and $\{d, e\}$ have to be ignored, as they have been treated in the first case above. So, only $(2^n - 4)$ points x on $\{a, b\}$ outside $\mathcal{O}$ need to be considered. Thus a, b, c, d, e appear together $10.(2^n - 4).\lambda_4 = 10.(2^n - 4)$ times in the blocks of $\mathcal{D}$ in this case.

Altogether a, b, c, d, e appear

$$15.(2^{n-1} - 2) + 10.(2^n - 4) = 70(2^{n-2} - 1)$$

times in the blocks of $\mathcal{D}$. Thus $(\mathcal{O}, \mathcal{D})$ is a 5-$(2^n + 2, 8, 70(2^{n-2} - 1))$ design, as desired.
□

Remark 2.1 We should remark that the construction of 5-designs from hyperovals using indexing 5-$(2^{n-1} + 1, k, \lambda)$ designs with $k \geq 5$ will not work. The reason is as follows. In determining the index of the constructed design there is a further case that needs to be considered when $k \geq 5$: namely, fixing a point, say a from the 5 points a, b, c, d, e. There are $(2^n - 3)$ lines ℓ_a through a not containing any of the points b, c, d, e. The problem is that the number of intersections of ℓ_a with lines $\{b, c\}$, $\{b, d\}$, $\{b, e\}$, $\{c, d\}$, $\{c, e\}$, $\{d, e\}$, cannot be uniquely determined (the number can be 6 or less than 6 depending on the given points b, c, d, e).

3 4-designs from hyperovals

In this section we focus on the case $t = 4$. In contrast to case $t = 5$, we will prove that the construction of 4-designs by hyperovals works for indexing 4-designs with any given block size $k \geq 4$.

Theorem 3.1 *Suppose that there exists a* 4-$(2^{n-1} + 1, k, \lambda_4)$ *design for* $k \geq 4$ *and* $n \geq 4$. *Then there exists a* 4-$(2^n + 2, 2k, \Lambda)$ *design with*

$$\Lambda = \frac{2(2k - 1)(2k - 3)}{(k - 2)(k - 3)}(2^{n-1} - 1)(2^{n-2} - 1)\lambda_4.$$

Proof. For any point $x \in \mathcal{P} \setminus \mathcal{O}$, choose a 4-$(2^{n-1} + 1, k, \lambda_4)$ design as the indexing design on Π_x. Note that for the indexing design $(\Pi_x, \mathcal{D}_x)$ we have $\lambda_3 = \frac{(2^{n-1}-2)}{(k-3)}\lambda_4$, $\lambda_2 = \frac{\binom{2^{n-1}-1}{2}}{\binom{k-2}{2}}\lambda_4$.

Let $a, b, c, d \in \mathcal{O}$ be any 4 points. There are three cases to be considered.

- Fixing a pair of lines $\{a, b\}$ and $\{c, d\}$. There are 3 possible choices for such pairs from the 4 points a, b, c, d. Now lines $\{a, b\}$ and $\{c, d\}$ intersect at a unique point $x \in \mathcal{P} \setminus \mathcal{O}$. Thus, for this case, a, b, c, d appear together $3.\lambda_2 = \frac{3\binom{2^{n-1}-1}{2}}{\binom{k-2}{2}}\lambda_4$ times in the blocks of $\mathcal{D}$.

- Fixing a line $\{a,b\}$. There are 6 possible choices of such lines from a,b,c,d. There are (2^n-1) points on line $\{a,b\}$ outside $\mathcal{O}$. One of them as the intersection of $\{a,b\}$ with $\{c,d\}$ has to be ignored, as it has been treated in the first case above. So, (2^n-2) points $x \in \{a,b\} \setminus \mathcal{O}$ need to be considered. Thus, in this case, a,b,c,d appear together $6.(2^n-2).\lambda_3 = \frac{6(2^n-2)(2^{n-1}-2)}{(k-3)}\lambda_4$ times in the blocks of $\mathcal{D}$.

- Fixing a point a. By reason of symmetry there is one possible choice for this case. There are (2^n-2) lines ℓ_a through a not containing any of the points b,c,d. There are (2^n-1) points x on $\ell_a \setminus \mathcal{O}$. The 3 intersections of ℓ_a with lines $\{b,c\}$, $\{b,d\}$, $\{c,d\}$ have to be ignored, as they have been already treated in the second case above. So, only (2^n-4) points $x \in \ell_a \setminus \mathcal{O}$ need to be considered. Thus, in this case, a,b,c,d, appear $(2^n-2).(2^n-4).\lambda_4$ times in the blocks of $\mathcal{D}$.

Altogether a,b,c,d appear

$$\begin{aligned}\Lambda = & \frac{3\binom{2^{n-1}-1}{2}}{\binom{k-2}{2}}\lambda_4 + \frac{6(2^n-2)(2^{n-1}-2)}{(k-3)}\lambda_4 + (2^n-2)(2^n-4)\lambda_4 \\ = & \frac{2(2k-1)(2k-3)}{(k-2)(k-3)}(2^{n-1}-1)(2^{n-2}-1)\lambda_4\end{aligned}$$

times in the blocks of $\mathcal{D}$. Thus $(\mathcal{O},\mathcal{D})$ is a 4-$(2^n+2, 2k, \Lambda)$ design, as desired. $\square$

The following is an immediate corollary of Theorem 3.1, where the indexing design is the complete 4-$(2^{n-1}+1, k, \binom{2^{n-1}-3}{k-4})$ design.

Corollary 3.2 *For $k \geq 4$ and $n \geq 4$ there exists a 4-$(2^n+2, 2k, \Lambda)$ design with*

$$\Lambda = \frac{2(2k-1)(2k-3)}{(k-2)(k-3)}(2^{n-1}-1)(2^{n-2}-1)\binom{2^{n-1}-3}{k-4}.$$

As first examples of the application of Theorem 3.1, we consider the 4-(v,k,λ) designs constructed by Hubaut [5], with the following parameters:

- $v = 2^{n-1}+1$, $k = 2^m$, $\lambda = (2^m-3)\prod_{i=2}^{m-1}\frac{2^{n-1-i}-1}{2^{m-i}-1}$, $2 < m < n-1$;

- $v = 2^{n-1}+1$, $k = 2^m+1$, $\lambda = (2^m+1)\prod_{i=2}^{m-1}\frac{2^{n-1-i}-1}{2^{m-i}-1}$, $2 < m < n-1$, $m \nmid n-1$.

Theorem 3.1 with indexing designs as the Hubaut 4-designs gives the following result.

Theorem 3.3 *For $n \geq 5$ there exist 4-$(v, 2k, \Lambda)$ designs with the following parameters:*

- $v = 2^n+2$, $2k = 2^{m+1}$,

  $\Lambda = 2\frac{(2^{m+1}-1)(2^{m+1}-3)}{(2^m-2)(2^m-3)}(2^{n-1}-1)(2^{n-2}-1)(2^m-3)\prod_{i=2}^{m-1}\frac{2^{n-1-i}-1}{2^{m-i}-1}$, $2 < m < n-1$;

- $v = 2^n+2$, $2k = 2^{m+1}+2$,

  $\Lambda = 2\frac{(2^{m+1}+1)(2^{m+1}-1)}{(2^m-1)(2^m-2)}(2^{n-1}-1)(2^{n-2}-1)(2^m+1)\prod_{i=2}^{m-1}\frac{2^{n-1-i}-1}{2^{m-i}-1}$, $2 < m < n-1$, $m \nmid n-1$.

4 s-resolvability of t-designs from hyperovals

In this section we turn our attention to the case where the indexing t-designs are s-resolvable. In this case, we prove that the t-designs from hyperovals are also s-resolvable. We present a proof of this statement for 4-designs in Theorem 4.1.

Theorem 4.1 *Assume that the indexing* 4-$(2^{n-1}+1,k,\lambda)$ *design can be partitioned into* m s-$(2^{n-1}+1,k,\delta_s)$ *designs, for* $s=2$ *or* 3. *Then the constructed* 4-$(2^n+2,2k,\Lambda)$ *design from a hyperoval can also be partitioned into* m s-$(2^n+2,2k,\Delta_s)$ *designs. More precisely,*

- *if* $s=2$, *then* 2-$(2^n+2,2k,\Delta_2)=2$-$(2^n+2,2k,\frac{(2k-1)}{(k-1)}(2^n-1)2^{n-1}\delta_2)$;
- *if* $s=3$, *then* 3-$(2^n+2,2k,\Delta_3)=3$-$(2^n+2,2k,\frac{(2k-1)}{(k-2)}(2^n-1)(2^{n-1}-1)\delta_3)$.

Proof. We show that there is a partition of the constructed 4-design into s-designs. Let

$$(\Pi_x,\mathcal{D}_x^1),\ldots,(\Pi_x,\mathcal{D}_x^m),$$

denote a partition of the indexing 4-design into m s-designs, i.e. each $(\Pi_x,\mathcal{D}_x^i)$ is a 2-$(2^{n-1}+1,k,\delta_2)$ or 3-$(2^{n-1}+1,k,\delta_3)$ design, $i=1,\ldots,m$.

Case $s=2$. Note that for a 2-$(2^{n-1}+1,k,\delta_2)$ design we have $\delta_1=\frac{2^{n-1}}{(k-1)}\delta_2$.

Let $a,b\in\mathcal{O}$ be any two points. Two cases need to be considered.

- (I): Fixing the line $\{a,b\}$. Since there are (2^n-1) points $x\in\{a,b\}\setminus\mathcal{O}$, the points a,b appear $(2^n-1).\delta_1=(2^n-1).2^{n-1}\delta_2/(k-1)$ times in the blocks of $(\mathcal{O},\mathcal{D})$ corresponding to $(\Pi_x,\mathcal{D}_x^i)$, $i=1,\ldots,m$.
- (II): Fixing a point a. By reason of symmetry there is one possible choice. There are 2^n lines ℓ_a through a not containing b. There are (2^n-1) points on $\ell_a\setminus\mathcal{O}$ which need to be counted. Thus, in case (II), a,b appear

$$2^n.(2^n-1).\delta_2$$

times in the block of $(\mathcal{O},\mathcal{D})$ corresponding to $(\Pi_x,\mathcal{D}_x^i)$, $i=1,\ldots,m$.

Altogether a,b appear

$$\Delta_2=\frac{1}{(k-1)}(2^n-1).2^{n-1}\delta_2+2^n(2^n-1).\delta_2=\frac{(2k-1)}{(k-1)}(2^n-1)2^{n-1}\delta_2$$

times in the blocks of $(\mathcal{O},\mathcal{D})$ corresponding to $(\Pi_x,\mathcal{D}_x^i)$, $i=1,\ldots,m$.

Thus $(\mathcal{O},\mathcal{D})$ has a 2-resolution with m resolution classes consisting of 2-$(2^n+2,2k,\frac{(2k-1)}{(k-1)}(2^n-1)2^{n-1}\delta_2)$ designs.

Case $s=3$. Note that for a 3-$(2^{n-1}+1,k,\delta_3)$ design we have $\delta_2=\frac{(2^{n-1}-1)}{(k-2)}\delta_3$.

Let $a,b,c\in\mathcal{O}$ be any three points. Two cases need to be considered.

- (I): Fixing a line $\{a, b\}$. There are 3 possible choices for such a line. As there are $(2^n - 1)$ points $x \in \{a, b\} \setminus \mathcal{O}$, the points a, b, c appear together

$$3.(2^n - 1).\delta_2 = 3(2^n - 1)\frac{(2^{n-1} - 1)}{(k-2)}\delta_3$$

times in the blocks of $(\mathcal{O}, \mathcal{D})$ corresponding to $(\Pi_x, \mathcal{D}_x^i)$, $i = 1, \ldots, m$.

- (II): Fixing a point a. By reason of symmetry there is one possible choice. There are $(2^n - 1)$ lines ℓ_a through a not containing b or c. There are $(2^n - 2)$ points on $\ell_a \setminus \mathcal{O}$ which need to be considered, since the intersection of ℓ_a with line $\{b, c\}$ has to be ignored. Thus a, b, c appear

$$(2^n - 1).(2^n - 2).\delta_3$$

times in the blocks of $(\mathcal{O}, \mathcal{D})$ corresponding to $(\Pi_x, \mathcal{D}_x^i)$, $i = 1, \ldots, m$.

Altogether a, b, c appear

$$\begin{aligned}\Delta_3 &= 3(2^n - 1)\frac{(2^{n-1} - 1)}{(k-2)}\delta_3 + (2^n - 1)(2^n - 2)\delta_3 \\ &= \frac{(2k-1)}{(k-2)}(2^n - 1)(2^{n-1} - 1)\delta_3\end{aligned}$$

times in the blocks of $(\mathcal{O}, \mathcal{D})$ corresponding to $(\Pi_x, \mathcal{D}_x^i)$, $i = 1, \ldots, m$.

Thus $(\mathcal{O}, \mathcal{D})$ has a 3-resolution with m resolution classes consisting of 3-$(2^n + 2, 2k, \frac{(2k-1)}{(k-2)}(2^n - 1)(2^{n-1} - 1)\delta_3)$ designs. □

Remark 4.1 As we are interested in s-resolutions with $s \geq 2$, we only consider this case in Theorem 4.1 above. However, a similar result for $s = 1$ can be proved. More exactly, if the indexing design 4-$(2^{n-1} + 1, k, \lambda)$ can be described as a union of m disjoint 1-$(2^{n-1} + 1, k, \delta_1)$ designs, then the constructed 4-$(2^n + 2, 2k, \Lambda)$ design is a union of m disjoint

$$1 - (2^n + 2, 2k, \Delta_1) = 1 - (2^n + 2, 2k, (2^n + 1)(2^n - 1)\delta_1)$$

designs.

Theorem 4.1 is still valid for 5-designs constructed from hyperovals, i.e. if the indexing design is s-resolvable, then so is the constructed 5-design. The proof is similar. So, we record the following result for s-resolvable 5-designs without the detail of the proof.

Theorem 4.2 *1. The $5-(2^n+2, 6, 15)$ design $n \geq 5$, constructed from a hyperoval can be partitioned into $m = (2^{n-1} - 1)/\lambda_{\min}$ disjoint $2 - (2^n + 2, 6, 5.2^{n-2}(2^n - 1)\lambda_{\min})$ designs, where $\lambda_{\min}$ is the smallest value for which a $2-(2^{n-1}+1, 3, \lambda_{\min})$ design exists.*

2. *Assume that* $2^{n-1}+1 \equiv 0 \pmod 3$, $n \geq 4$. *Then the* $5-(2^n+2, 8, 70(2^{n-2}-1))$ *design in Theorem 2.1 can be partitioned into* $m = (2^{n-1}-2)/\lambda_{\min}$ *disjoint* $3-(2^n+2, 8, \frac{7}{2}.(2^n-1)(2^{n-1}-1)\lambda_{\min})$ *designs, where* $\lambda_{\min}$ *is the smallest value for which a* $3-(2^{n-1}+1, 4, \lambda_{\min})$ *design exists.*

Proof.

1. Using a large set $LS_{\lambda_{\min}}(2, 3, 2^{n-1}+1)$ of the indexing design [7], i.e. the indexing design can be partitioned into $m = (2^{n-1}-1)/\lambda_{\min}$ disjoint 2-$(2^{n-1}+1, 3, \lambda_{\min})$ designs.

2. If $2^{n-1}+1 \equiv 0 \pmod 3$, then using a large set $LS_{\lambda_{\min}}(3, 4, 2^{n-1}+1)$ of the indexing design [7], i.e. the indexing design can be partitioned into $m = (2^{n-1}-2)/\lambda_{\min}$ disjoint 3-$(2^{n-1}+1, 4, \lambda_{\min})$ designs.

□

In the following we look at some specific families of 3-resolvable 4-designs as consequences of Theorems 3.1 and 4.1. It is known that the simple 4-$(2^{n-1}+1, 5, 5)$ design with $(n-1) \geq 5$ odd, constructed by Alltop [1] is 3-resolvable with $m = (2^{n-1}-2)/6$ resolution classes [10, 11]. Each class is a 3-$(2^{n-1}+1, 5, 15)$ design. Thus, when the indexing design is the Alltop design, we have the following result from Theorems 3.1 and 4.1.

Theorem 4.3 *There exists a* 4-$(2^n+2, 10, 105(2^{n-1}-1)(2^{n-2}-1))$ *design for even* $n \geq 6$. *The design can be partitioned into* $(2^{n-1}-2)/6$ *3-designs with parameters* $3-(2^n+2, 10, 45(2^n-1)(2^{n-1}-1))$.

In a series of papers [2, 3, 4] Bierbrauer constructed several interesting infinite families of simple 4-designs for $k = 6, 8, 9$ with constant indices. The construction makes use of the projective general linear group $\mathrm{PGL}(2, q)$, $q = 2^n$ which has a sharply 3-transitive action on the projective line $X = \mathrm{GF}(q) \cup \{\infty\}$. Here are some parameters of these designs: 4-$(2^{n-1}+1, 6, \lambda_4)$, $\lambda = 10, 60, 70$, 4-$(2^{n-1}+1, 8, 35)$, 4-$(2^{n-1}+1, 9, 84)$. It turns out, as noted in [11], that all here mentioned 4-designs are 3-resolvable with $m = (q-2)/6$ resolution classes. In detail, the 4-$(2^{n-1}+1, 6, \lambda_4)$ design is partitioned into 3-$(2^{n-1}+1, 6, 2\lambda_4)$ designs; the 4-$(2^{n-1}+1, 8, 35)$ design into 3-$(2^{n-1}+1, 8, 42)$ designs and the 4-$(2^{n-1}+1, 9, 84)$ design into 3-$(2^{n-1}+1, 9, 84)$ designs. Hence, from Theorems 3.1 and 4.1 we have the following result.

Theorem 4.4 *Let* n *be an even integer with* $n \geq 6$ *and* $\gcd(n-1, 6) = 1$. *Then the following 3-resolvable 4-designs exist.*

1. *A* 4-$(2^n+2, 12, \frac{33}{2}\lambda_4(2^{n-1}-1)(2^{n-2}-1))$ *design,* $\lambda_4 = 10, 60, 70$. *The design can be partitioned into* $(2^{n-1}-2)/6$ *3-designs with parameters* $3-(2^n+2, 12, \frac{11}{2}\lambda_4(2^n-1)(2^{n-1}-1))$.

2. *A* 4-$(2^n+2, 16, 455(2^{n-1}-1)(2^{n-2}-1))$ *design. The design can be partitioned into* $(2^{n-2}-2)/6$ *3-designs with parameters* $3-(2^n+2, 16, 105(2^n-1)(2^{n-1}-1))$.

3. *A* 4-$(2^n+2, 18, 1020(2^{n-1}-1)(2^{n-2}-1))$ *design. The design can be partitioned into* $(2^{n-1}-2)/6$ *3-designs with parameters* $3-(2^n+2, 18, 204(2^n-1)(2^{n-1}-1))$.

Remark 4.2 It is worth remarking that Corollary 3.2 will provide infinitely many s-resolvable 4-designs. This is the case when the complete indexing designs have a large set of s-designs with $s = 2$ or 3.

5 Simplicity or non-simplicity of designs from hyperovals

The question regarding whether the designs constructed from hyperovals are simple or non-simple appears to depend on three components: the underlying projective planes, the types of hyperovals and the indexing designs. By using regular hyperovals in $\mathrm{PG}(2, 2^n)$ the Lonz-Vanstone 5-$(2^n+2, 6, 15)$ designs are non-simple, as mentioned in the introduction. Nevertheless it is not known if the Lonz-Vanstone designs are still non-simple when other types of hyperovals are used. It seems to be possible to check it with $\mathrm{PG}(2, 16)$ (there are exactly two types of hyperovals) and with $\mathrm{PG}(2, 32)$ (exactly six types of hyperovals), see [9] for example. However, if we focus on hyperoval 4-designs based on $\mathrm{PG}(2, 32)$, much more work needs to be done, since there are many more indexing 4-$(17, k, \lambda)$ designs to be considered. Thus, in this direction finding an answer to the question seems to be difficult because indexing designs are innumerable, and hyperovals and projective planes of order 2^n for $n \geq 5$ are still far away from being classified. On the other side, when k is not small there might exist a combinatorial argument to show the simplicity of the hyperoval designs without employing the structure of the involved components. Moreover, since the construction by hyperovals produces designs in abundance, we believe that the following conjecture should be true.

Conjecture: There exist simple designs from hyperovals.

6 Conclusion

We have studied a construction of t-designs and s-resolvable t-designs by using hyperovals in projective planes of order 2^n. It is proven that 4- or 5-designs can be constructed and particularly there exist infinitely many 4-designs with any even block size. The question of simplicity or non-simplicity of designs from hyperovals remains an open problem.

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