SCHRIFTENREIHE DER FAKULTÄT FÜR MATHEMATIK

A Note on Stability for Risk Averse Stochastic Complementarity Problems

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Abstract Introduced by Chen and Fukushima in 2005, expected residual minimization (ERM) has become an established approach to complementarity problems under stochastic uncertainty. NCP and merit functions allow to recast deterministic complementarity problems as optimization problems, where the objective function is the total residual. Based on this reformulation, the risk neutral ERM formulation aims to minimize the expectation of the total residual. In the present work, we propose an extension of the ERM formulation by replacing the expectation with a more general convex, nondecreasing and law-invariant risk functional. Our model allows to take risk aversion into account. We examine joint continuity of the objective function with respect to both, the decision and the probability measure induced by the entering random vector. The latter allows to investigate the problem's behavior when working with approximations of the original distribution. We consider perturbations of the underlying probability measure with respect to the weak topology and derive a stability of the optimal value function and the optimal solution set mapping. The paper concludes with demonstrating that our assumptions hold for almost all practically relevant NCP functions.

Keywords stochastic complementarity problem · NCP function · expected residual minimization · risk aversion · stability

Mathematics Subject Classification (2000) 90C15 · 90C31 · 90C33

1 Introduction

Fix a continuous self-map F of the n dimensional Euclidian space. The general complementarity problem is to decide whether there exists a nonnegative vector x that is orthogonal to its nonnegative image $F(x)$. Such problems arise in the context of classical KKT conditions, special cases of equilibrium problems or reformulations of different optimization problems and have wide-spread applications in the fields of engineering, economy, game theory and various other mathematical domains (see e.g. section 1 in [1], section 1.5 in [2], [3]).

Many real-world applications have to cope with unknown data. A special case arises if the data uncertainty is stochastic and purely exogenous. The latter means that the random data does not depend on any of the decisions to be made. A stochastic complementarity problem arises, when the function F also depends on a random parameter $z(\omega)$. Applications of these problems include Wardrop's user equilibrium, traffic equilibrium problems and the pricing of American options (see e.g. [4,5,6]). One approach for handling such problems proceeds by replacing the random function

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F with its expectation and is referred to as expected value (EV) formulation (see e.g. [7,8], section 3.1.1 in [5], section 2.2 in [6]).

In the present work we take a different route. Although complementarity problems are feasibility problems by nature, so called NCP functions allow to reformulate them as minimization problems. Involving stochastic data uncertainty, this approach has lead to the expected residual minimization (ERM) formulation $\min \mathbf{E}[\Phi(x,\omega)]$ introduced by Chen and Fukushima (2005) [9]. Inspired by structural similarities to one-stage stochastic optimization we propose a model allowing to take into account risk aversion by adding a quantification of risk to the objective function of the ERM model. The issue of stability arises from perturbations of the underlying distribution (or the induced Borel probability measure), e.g. originating from approximation. For this purpose, we consider the dependence of the objective function Q on both x and the entering measure. To our knowledge, the stability of these models has not been examined in the context of stochastic complementarity programming.

In this paper we endow the space of Borel probability measures with the topology of weak convergence and derive weak continuity of a restriction of Q to an appropriate subset. This result is based on [10] under a growth condition imposed on function F as well as the entering NCP function and the assumption that the considered risk measure is convex, nondecreasing and law-invariant. In particular, we cover all convex (and hence all coherent) risk measures including the expected exceedance of a given target level, the semideviation and the conditional value-at-risk. Stability of optimal values and solution sets then follows from standard results.

This paper is organized as follows: In section 2, we describe the structure of the underlying deterministic model to be used. Stability analysis is carried out in section 3. Finally, we show that our assumptions are fulfilled for a large number of well known NCP functions.

2 Complementarity Problems under Stochastic Uncertainty

Our starting point is a parametric version of the standard complementarity problem (e.g. on p. 4 in [2]): The *parametric complementarity problem* on $X \subseteq \mathbb{R}^n$ is to find a vector $x \in X$ (or to prove its nonexistence) such that

$$\mathrm{CP}(X, F(\cdot, z)) \qquad x \geq 0,\ F(x,z) \geq 0,\ x^\top F(x,z) = 0, \tag{1}$$

where $F : X \times \mathbb{R}^s \to \mathbb{R}^n$ is a continuous function. Since $x \geq 0$ is part of (1), we may assume $X \subseteq \mathbb{R}^n_+$ without loss of generality. Throughout the paper, we fix X and F and we can write $\mathrm{CP}(z)$ instead of $\mathrm{CP}(X, F(\cdot, z))$.

The complementarity problem $\mathrm{CP}(z)$ consists of a system of inequalities and an equation. The properties of so called NCP functions (e.g. in section 1.5.1 in [2]) can be used to reformulate this system into a system of equations:

Definition 2.1 (NCP function)
A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is called a *NCP function* (or a *C-function*), if for any pair $(a,b)^\top \in \mathbb{R}^2$

$$\phi(a,b) = 0 \Leftrightarrow (a \geq 0,\ b \geq 0,\ ab = 0)$$

is fulfilled.

Among the most established examples of NCP functions are the *Fischer-Burmeister function* and the *minimum function* (or *natural residual*). For details and a comprehensive overview of other NCP functions we refer to section 1.5.1 in [2] or the appendix of [11]. Certain properties of specific NCP functions shall be addressed in section 4. Throughout section 2 and 3 of the present work, we fix an arbitrary NCP function ϕ.

Let F_j denote the $j-$th component function of F. The complementarity problem $\mathrm{CP}(z)$ can be transformed to a system of n nonlinear equations:

$$\phi(F_j(x,z), x_j) = 0 \quad \text{for all } j \in \{1, \ldots, n\} \text{ and } x \in X. \tag{2}$$

We proceed by summarizing the n equations in (2) to a single expression by means of a suitably chosen merit function (see e.g. section 1.5.3 in [2], on p. 806 in [11]). For the present analysis, we use a vectorvalued function to rewrite (2) as

$$\Phi(x,z) := (\phi(F_1(x,z),x_1),\ldots,\phi(F_n(x,z),x_n)^\top = 0 \quad \text{and } x \in X. \tag{3}$$

and employ the so called naturally associated merit function: the squared Euclidean norm $\theta(x) := \|\Phi(x,z)\|^2$. The complementarity problem CP(z) or the problems (2) and (3), respectively, are equivalent to the following minimization problem (e.g. on p.52 in [7]):

$$\min_{x\in X} \ \|\Phi(x,z)\|^2. \tag{4}$$

Like every merit function, θ is bounded from below by zero. Hence, x is a solution of CP(z) iff x is an optimal solution to (4) that yields the value 0.

Based on these considerations, we now turn our attention to the aspect of uncertainty in complementarity problems. Let $z(\cdot) : \Omega \to \mathbb{R}^s$ be a random vector on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The stochastic complementarity problem arsises from considering the problems $C(z(\omega))$, $\omega \in \Omega$:

The *stochastic complementarity problem* is to find a vector $x \in X$ (or to prove its nonexistence) such that

$$\text{SCP}(F) \qquad x \geq 0,\ F(x,z(\omega)) \geq 0,\ x^\top F(x,z(\omega)) = 0 \quad \mathbb{P}\text{-almost surely,}$$

where $F : X \times \mathbb{R}^s \to \mathbb{R}^n$ is a continuous function.

The vector $x \in X$ must be selected under uncertainty, i.e. the outcome of the random variable $z(\omega)$ is not known before the decision of x. This restriction is an *information* or *nonanticipativity constraint*. In addition, the choice of x may not influence the distribution of $z(\cdot)$, i.e. the stochasticity is assumed to be purely exogenous.

Remark 2.1 The above problem is robust in the sense that a solution x of SCP(F) has to be feasible for CP($z(\omega)$) for all possible realizations of the unknown parameter. Hence, it can be understood as deciding the feasibility of a possibly infinite system of complementarity problems. Note that a SCP(F) may be infeasible even if each of the individual deterministic problems is solvable. Furthermore, a SCP(F) does scarcely depend on the probability of the individual realizations, which may lead to a highly unlikely scenario causing its infeasibility.

Remark 2.2 If $z(\omega) = z \in \mathbb{R}^s$ is constant almost surely, the SCP(F) reduces to the standard CP(z). This is not the case in general, i.e. F is a random vector $F(x, z(\omega))$.

Building on these observations, we aim to select an 'optimal' vector $x \in X$ using a deterministic model employing risk measures. The model relates to the equivalent reformulation in (4) of CP(z) and is based on the expected residual minimization (ERM) method (see e.g. [7,8,9]):

$$(ERM) \qquad \min_{x\in X} \ \mathbf{E}[\|\Phi(x,z(\omega))\|^2],$$

where $\mathbf{E}$ denotes the expectation with respect to ω. This model was first proposed by Chen and Fukushima in 2005 in [9] and a well-posed optimization problem that is not convex in general. In contradistinction to the stochastic complementarity problem SCP(F), the decision on x is now entirely based on minimizing the expected values of the random variables in the family $\{\|\Phi(x,z(\omega))\|^2 :\ x \in X\}$.

A more general model is formed by replacing the expected value $\mathbf{E}$ with a mapping $\mathcal{R}$ from the space of random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ to the extended reals $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. Chosing $\mathcal{R}$ as a weighted sum of the expectation and some quantification of risk now allows to take into account risk aversion:

$$\min_{x\in X} \ \mathcal{R}[\|\Phi(x,z(\omega)\|^2]. \tag{5}$$

Remark 2.3 If $\mathcal{R}[\mathcal{Y}] = 0$ is equivalent to $\mathbb{P}\{\mathcal{Y}(\omega) = 0\} = 1$, problem (5) yields the optimal value zero iff each of its optimal solutions solves SCP(F).

3 On Stability of Mean Risk Formulations

In this section, we examine stability of problem (5) under perturbations of the underlying Borel probability measure μ induced by the random parameter $z(\omega)$. Equip the space of Borel probability measures $\mathcal{P}(\mathbb{R}^s)$ with the topology of weak convergence (see e.g. [12] for a discussion). The following example shows that problem (5) can be highly instable:

Example 3.1 Consider the case where $X = \mathbb{R}_{\geq 0}$, $F(x,z) = z$ and the NCP function is defined by

$$\phi(a,b) := \begin{cases} 0, & \text{if } ab = 0,\ a \geq 0,\ b \geq 0 \\ \lambda, & \text{else} \end{cases}$$

and a real parameter $\lambda \neq 0$. Furthermore, assume that $\mathcal{R}$ is the expectation and the Borel probability measure μ induced by the random parameter $z(\omega)$ is the Dirac measure at 0 (denoted by δ_0). Then the optimal value of problem (5) is zero and every $x \in X$ is an optimal solution.

For every neighborhood $\mathcal{N}$ of δ_0 with respect to the topology of weak convergence there is a real number $\epsilon \neq 0$ such that $\delta_\epsilon \in \mathcal{N}$. However, the solution of (5) with respect to z_ϵ induced by δ_ϵ depends on the signum of ϵ: For $\epsilon < 0$, every $x \in X$ is optimal with value λ^2, while $\epsilon > 0$ yields the unique optimal solution $x = 0$. The reason for this instability is the discontinuity of NCP function ϕ.

First we derive joint continuity of the objective function with respect to the decision x and the parameter μ. For a comprehensive class of risk measures, it has been shown that the desired continuity is implied by a growth condition imposed on the underlying deterministic model (see [10]). We work with the following assumptions:

(B_Φ) The function Φ is Borel measurable.
(G_Φ) There exist a constant $\gamma_\Phi \geq 0$ and a locally bounded mapping $\eta_\Phi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, i.e. convergence of $\{x_n\}_{n\in\mathbb{N}}$ implies boundedness of $\{\eta_\Phi(x_n)\}_{n\in\mathbb{N}}$, such that

$$\|\Phi(x,z)\| \leq \eta_\Phi(x)(\|z\|^{\gamma_\Phi} + 1) \quad \text{for all } (x,z) \in \mathbb{R}^n \times \mathbb{R}^s.$$

Lemma 3.1 (Sufficient conditions for (B_Φ) and (G_Φ))
Assume that the following statements hold true:

(B_F) *The function F is Borel measurable.*
(G_F) *There exist a constant $\gamma_F \geq 0$ and a locally bounded mapping $\eta_F : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that*

$$\|F(x,z)\| \leq \eta_F(x)(\|z\|^{\gamma_F} + 1) \quad \textit{for all } (x,z) \in \mathbb{R}^n \times \mathbb{R}^s.$$

(B_ϕ) *The NCP function ϕ is Borel measurable.*
(G_ϕ) *There exist a constant $\gamma_\phi \geq 0$ and a locally bounded mapping $\eta_\phi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that*

$$\|\phi(a,b)\| \leq \eta_\phi(b)(|a|^{\gamma_\phi} + 1) \quad \textit{for all } (a,b) \in \mathbb{R}^2.$$

Then (B_Φ) and (G_Φ) hold with $\gamma_\Phi = \gamma_F\gamma_\phi$.

Proof By (B_F) and (B_ϕ), function Φ is Borel measurable as a composition of Borel measurable functions. Furthermore, for arbitrary $(x,z) \in \mathbb{R}^n \times \mathbb{R}^s$ we have

$$\begin{aligned} \|\Phi(x,z)\| &\leq n \max_{j=1,\dots,n} |\phi(e_j^\top F(x,z), e_j^\top x)| \\ &\underset{(G_\phi)}{\leq} n \max_{j=1,\dots,n} \eta_\phi(e_j^\top x)(|e_j^\top F(x,z)|^{\gamma_\phi} + 1) \\ &\leq n(\|F(x,z)\|^{\gamma_\phi} + 1) \max_{j=1,\dots,n} \eta_\phi(e_j^\top x) \\ &\underset{(G_F)}{\leq} n(\eta_F(x)^{\gamma_\phi}(\|z\|^{\gamma_F} + 1)^{\gamma_\phi} + 1) \max_{j=1,\dots,n} \eta_\phi(e_j^\top x) \\ &\leq \eta_\Phi(x)(\|z\|^{\gamma_F\gamma_\phi} + 1), \end{aligned}$$

where $\eta_\Phi(x) := n \max_{j=1,\dots,n} \eta_\phi(e_j^\top x)((2\eta_F(x))^{\gamma_\phi} + 1)$ is locally bounded. Hence, (G_Φ) holds with $\gamma_\Phi = \gamma_F\gamma_\phi$. □

Remark 3.1 (B_ϕ) does not hold for every NCP function in the sense of Definition 1 in [11]: Let $\chi_{\mathcal{C}}$ denote the indicator function of a set $\mathcal{C} \subset \mathbb{R}^2$ that is not Borel measurable (see [13] for examples). Then $\phi_{CEB} : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\phi_{CEB}(a,b) := \begin{cases} 0, & \text{if } ab = 0,\ a \geq 0,\ b \geq 0 \\ \chi_{\mathcal{C}}(a,b) + 1, & \text{else} \end{cases}$$

is not Borel measurable.

Remark 3.2 (G_ϕ) does not hold for every NCP function either: The mapping $\phi_{CEG} : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\phi_{CEG}(a,b) := \begin{cases} 0, & \text{if } ab = 0,\ a \geq 0,\ b \geq 0 \\ \exp(a), & \text{else} \end{cases}$$

yields a counterexample.

In the following analysis we confine ourselves to a special class of risk measures that are induced by a mapping $\rho : L^p(\Omega', \mathcal{A}', \mathbb{P}') \to \mathbb{R}$, where $p \geq 1$ and $(\Omega', \mathcal{A}', \mathbb{P}')$ is an atomless probability space. We assume that ρ is convex, nondecreasing w.r.t. the $\mathbb{P}'$-almost sure partial order and law invariant. Such a mapping gives rise to a function

$$\mathcal{R}_\rho : \mathcal{M}_1^p := \{\nu \in \mathcal{P}(\mathbb{R}) : \int_{\mathbb{R}} |t|^p \nu(dt) < \infty\} \to \mathbb{R}$$

via $\mathcal{R}_\rho(\nu) := \rho(F_\nu^{\leftarrow}(U))$. Here, $F_\nu^{\leftarrow}$ denotes the left-continuous quantile function of the distribution function of ν and U is uniformly distributed on the open unit interval.

Remark 3.3 Every convex or coherent risk measure (see [14] or [15] for discussions) yields a possible choice for ρ. In particular, the approach covers the expectation, the upper semideviation and the conditional value-at-risk as well as every conic combination of covered functionals.

Setting $f(x,z) := \|\Phi(x,z)\|^2$ then allows to reformulate problem (5) as

$$(P(\mu)) \qquad \min_{x \in X} \underbrace{\mathcal{R}_\rho[(\delta_x \otimes \mu) \circ f^{-1}]}_{=:Q(x,\mu)}, \tag{6}$$

where δ_x denotes the Dirac measure at x. Assume (B_Φ) and (G_Φ) and set $\mathcal{M}_s^{2p\gamma_\Phi} := \{\nu \in \mathcal{P}(\mathbb{R}^s) : \int_{\mathbb{R}} \|t\|^{2p\gamma_\Phi} \nu(dt) < \infty\}$. A straightforward calculation then shows that Q is well defined and real-valued on $\mathbb{R}^n \times \mathcal{M}_s^{2p\gamma_\Phi}$ (see Lemma 2.1 in [10]).

A subset $\mathcal{M}$ of $\mathcal{M}_s^{2p\gamma_\Phi}$ is called *locally uniformly* $\|\cdot\|^{2p\gamma_\Phi}$−integrating, if for any $\nu \in \mathcal{M}$ there exists some open neighborhood $\mathcal{N}$ of ν with respect to the topology of weak convergence such that

$$\lim_{a \to \infty} \sup_{\mu \in \mathcal{N} \cap \mathcal{M}} \int_{\mathbb{R}^s} \|z\|^{2p\gamma_\Phi} \cdot 1_{]a,\infty[}(\|z\|^{2p\gamma_\Phi})\ \mu(dz) = 0.$$

For equivalent characterizations and examples of such sets we refer to [16].

Theorem 3.1 (Weak Continuity of $Q|_{\mathbb{R}^n \times \mathcal{M}}$)
Let $\mathcal{M} \subseteq \mathcal{M}_s^{2p\gamma_\Phi}$ be a locally uniformly $\|\cdot\|^{2p\gamma_\Phi}$−integrating set and let D_Φ denote the set of discontinuities of function Φ. Assume (B_Φ), (G_Φ) and that $(x,\mu) \in \mathbb{R}^n \times \mathcal{M}$ is such that $(\delta_x \otimes \mu)(D_\Phi) = 0$ holds. Then $Q|_{\mathbb{R}^n \times \mathcal{M}}$ is continuous at (x,μ) with respect to the product topology of the standard topology on $\mathbb{R}^n$ and the relative topology of weak convergence on $\mathcal{M}$.

Proof Let D_f denote the set of discontinuities of $f(x,z) = \|\Phi(x,z)\|^2$. We have $D_f \subseteq D_\Phi$ and hence $(\delta_x \otimes \mu)(D_f) = 0$. Furthermore, function f is Borel measurable by (B_Φ) and (G_Φ) implies

$$|f(x,z)| \leq 4\eta_\Phi(x)^2(\|z\|^{2\gamma_F} + 1) \quad \text{for all } (x,z) \in \mathbb{R}^n \times \mathbb{R}^s.$$

Hence, Theorem 2.2 in [10] is applicable and yields the desired continuity. □

The following corollary points out a sufficient condition for $(\delta_x \otimes \mu)(D_f) = 0$ to hold globally:

Corollary 3.1 *In addition to the assumptions of Theorem 3.1, let function Φ be continuous. Then $Q|_{\mathbb{R}^n \times \mathcal{M}}$ is continuous on $\mathbb{R}^n \times \mathcal{M}$ with respect to the product topology of the standard topology on $\mathbb{R}^n$ and the relative topology of weak convergence on $\mathcal{M}$.*

Proof The continuity of function Φ implies $D_f = \emptyset$. □

Remark 3.4 In particular, function Φ is continuous if both function F and NCP function ϕ are continuous.

Now we turn our attention to the stability of problem $(P(\mu))$:
Let $\varphi : \mathcal{P} \to \mathbb{R} \cup \{\pm\infty\}$ with $\varphi(\mu) := \inf_{x \in X} Q(x, \mu)$ and $\psi : \mathcal{P} \to 2^{\mathbb{R}^n}$ with $\psi(\mu) := \mathrm{argmin}_{x \in X} Q(x, \mu)$ denote the optimal value function and the optimal solution set mapping of problem (6).

Corollary 3.2 *Let the assumptions of Corollary 3.1 be fulfilled. Then $\varphi|_{\mathcal{M}}$ is upper semicontinuous on $\mathcal{M}$ with respect to the relative topology of weak convergence.*

Proof $Q|_{X \times \mathcal{M}}$ is continuous on $X \times \mathcal{M}$ and the feasible set X is fixed. By section 4.1 in [17] that yields the upper semicontinuity of $\varphi|_{\mathcal{M}}$. □

Corollary 3.3 (Stability of $(P(\mu))$)
Let the assumptions of Corollary 3.1 be fulfilled and assume that X is compact. Then $\varphi|_{\mathcal{M}}$ is continuous and $\psi|_{\mathcal{M}}$ is upper semicontinuous on $\mathcal{M}$ with respect to the relative topology of weak convergence.

Proof See Proposition 1.1 in [18]. □

4 Special NCP Functions

In this section, we show that the assumptions (B_ϕ) and (G_ϕ) posed on NCP function ϕ in Lemma 3.1 are fulfilled for the vast majority of all practically relevant NCP functions. Hence, the crucial assumptions (B_Φ) and (G_Φ) hold whenever the underlying complementarity problem is such that (B_F) and (G_F) are fulfilled.

Throughout the following analysis, we call a function $g : \mathbb{R}^2 \to \mathbb{R}$ *growing with respect to the exponent* $\gamma_g \geq 0$, if there exists a locally bounded mapping $\eta_g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that

$$|g(a, b)| \leq \eta_g(b)(|a|^{\gamma_g} + 1) \quad \text{for all } (a, b) \in \mathbb{R}^2.$$

Lemma 4.1 *Let $g, h : \mathbb{R}^2 \to \mathbb{R}$ be growing with respect to some exponents $\gamma_g, \gamma_h \geq 0$, respectively. Then the following statements hold true:*

(1) $|g|$ is growing with respect to γ_g.
(2) For $q \geq 0$, g^q is growing with respect to $q\gamma_g$.
(3) For real constants α and β, $\alpha g + \beta h$ is growing with respect to

$$\max\{(1 - \chi_{\{0\}}(\alpha))\gamma_g, (1 - \chi_{\{0\}}(\beta))\gamma_h\}.$$

(4) gh is growing with respect to $\gamma_g + \gamma_h$.

Proof The proof of (1) and (2) is straightforward. For (3) and (4), distinguish between the cases $|a| \geq 1$ and $|a| < 1$ and consider the locally bounded mappings $\eta_{\alpha g + \beta h} := 2(\alpha \eta_g + \beta \eta_h)$ and $\eta_{gh} := 4\eta_g \eta_h$. □

Remark 4.1

(a) The mappings $(a, b) \mapsto a$, $(a, b) \mapsto b$ and all indicator functions are growing with respect to 1, 0 and 0, respectively.
(b) Furthermore, polyhedral sets are Borel measurable and Borel measurability is preserved under the transformations of Lemma 4.1. Hence, the result allows to verify assumptions (B_ϕ) and (G_ϕ) for all of the 31 special NCP functions given in the Appendix of [11].
(c) For NCP function proposed by Ulbrich (see [19]), the following estimate is helpful: For $0 < |a| + |b| \leq \kappa$, we have

$$\left| \frac{ab}{\kappa(1 - exp\left(\frac{|a|+|b|}{\kappa}\right))} \right| \leq \frac{|ab|}{|a| + |b|} \leq \kappa,$$

which yields the desired growth. In view of Remark 3.4, it is also important to point out that all of these functions are continuous.

Another important class of NCP functions is given by Mangasarians family of NCP functions (see e.g. [20]) defined by

$$\{\phi_\zeta(a,b) = \zeta(|a-b|) - \zeta(b) - \zeta(a) : \zeta : \mathbb{R} \to \mathbb{R} \text{ is strictly increasing and } \zeta(0) = 0\}.$$

By the monotonicity of ζ, all elements of the above family are Borel measurable, while (G_ϕ) does not hold in general: $\zeta(a) = \exp(a) - 1$ yields a counterexample. However, if there exist constants $\gamma_\zeta, C \geq 0$ such that $|\zeta(a)| \leq |a|^{\gamma_\zeta} + C$ holds for all $a \in \mathbb{R}$, then the resulting NCP function ϕ_ζ is growing with respect to γ_ζ.

5 Conclusions

In the present work, we have proposed an extension of the ERM formulation for complementarity problems under stochastic uncertainty that works with more general objectives than the expectation. Special cases of our model include objective functions based on convex risk measures like the Expected Exceedance and the Conditional Value-at-Risk and allow to take into account risk aversion. Our main results states that the arising objective functions are jointly continuous with respect to both the decision variable and the underlying probability measure (with respect to the topology of weak convergence). This result allows to derive a stable behavior of the optimal value function and the optimal solution set mapping. Finally, we have shown that the assumptions on the underlying deterministic model hold for almost all NCP functions of practical relevance.

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