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WEAK CONTINUITY OF RISK FUNCTIONALS WITH APPLICATIONS TO STOCHASTIC PROGRAMMING

by

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Abstract. Measuring and managing risk has become crucial in modern decision making under stochastic uncertainty. In two-stage stochastic programming, mean risk models are essentially defined by a parametric recourse problem and a quantification of risk. From the perspective of qualitative robustness theory, we discuss sufficient conditions for continuity of the resulting objective functions with respect to perturbation of the underlying probability measure. Our approach covers a fairly comprehensive class of both stochastic-programming related risk measures and relevant recourse models. Not only this unifies previous approaches but also extends known stability results for two-stage stochastic programs to models with mixed-integer convex recourse and quadratic integer recourse, respectively.

Key words. Stochastic programming, mean risk models, stability, risk functionals

AMS subject classifications. 90C15, 91B30

1. Introduction.

Since the last decade risk management has become an important issue from a practical view point, and as a research field as well. The interests range from pragmatic solutions for practitioners to research which has founded a hybrid of a new mathematical discipline integrating several fields such as stochastics (e.g.[22], [32]), optimization (e.g. [24], [38]), numerical analysis (e.g. [12]) and, when integer variables occur, also algebra and discrete mathematics (e.g. [36]). Since economic risks like credits, prices of stocks or insurance claims are typically faced with uncertainty, most of the methods are settled within a stochastic framework representing risks in terms of random variables. Then basic objects are often stochastic functionals, i.e. real-valued functions defined on sets of random variables expressing economic risks. As a prominent example the so called coherent risk measures may be pointed out. This concept was introduced in [2] as a mathematical tool to assess the risks of financial positions. They are building blocks in quantitative risk management (see [22], [24], [32]), and they have been suggested as a systematic approach for calculations of insurance premia (cf. [15]). Besides the ordinary expectation, the conditional value at risk and the upper semideviation are the most known examples for coherent risk measures. However, meanwhile the more general notion of convex risk measure has replaced coherent risk measures in playing their roles.

Of particular interest are stochastic functionals which are distribution invariant, identifying risks with identical distributions. For instance the expectation, the conditional value at risk and the upper semideviation satisfy this property. They all may be redefined as functionals on sets of probability measures representing the distributions of the risks. Recent contributions analyze analytic properties of such functionals like specific types of continuity and differentiability (cf. [18], [19]). Such properties have immediate applications for statistical issues of the functionals e.g. the sample average approximation method (SAA) ([18], [19] again; see also [24], [38], [6], [32]). The aim of the present paper is to point out how investigations of stochastic programming problems may profit from continuity properties of distribution invariant stochastic functionals which are used for objective functions. In technical terms, investigations

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are devoted to topological considerations on spaces of random variables induced in a natural way by stochastic programming problems.

Stochastic programming is based on the crucial assumption that uncertainty can be captured by a probability measure which, in turn, has impact on structural and/or algorithmic properties of the objective function and/or the constraints. The probability measure usually being subjective or resulting from statistical estimation the issue of stability comes to the fore, i.e., small perturbations of the measure shall lead to only small perturbations of the optimal value and the optimal solution sets. Based on arguments from [7] this is made mathematically rigorous by appealing to parametric optimization and aiming at (semi-)continuity of optimal-value functions and of multifunctions given by sets of optimal solutions. Typically, the parameter spaces may vary from Euclidean spaces of parameters of distributions to topological or metric spaces of probability measures equipped with weak convergence of probability measures [16, 28] or with suitable probability metrics [31, 26, 29]. In classical stochastic programming the objective function is described in terms of expectations representing a risk neutral attitude of the decision maker. More recent contributions try to incorporate risk aversion of the decision makers using different distribution invariant coherent risk measures, where also both, continuous or mixed-integer variables are involved (cf. e.g.[30, 21, 37, 38]).

In the nutshell stability of stochastic programming refers to continuity of the distribution invariant functionals used for the objective functions. In the above mentioned literature this viewpoint has not been exploited systematically. Instead by individual reasoning suitable settings of uniform integrability or moment conditions had to be adapted to the individual objectives (starting with [16, 28, 31]), and sufficient conditions had to be found to transfer weak convergence of sequences of probability measures to sequences of image measures (see [35] for an example of detailed elaboration).

The present paper is an attempt of systemization. We provide an umbrella for most of the different settings in the papers referred to in the previous paragraphs. The line of reasoning is inspired by recent investigations on continuity of distribution invariant convex risk measures ([18]). They are based on the so called $\psi-$weak topologies which is a quite new class of topologies for sets of probability measures enclosing the topology of weak convergence. We shall extend the studies to more general distribution invariant stochastic functionals which will be referred to as risk functionals.

More precisely, we introduce general risk functionals living on sets of probability measures satisfying some moment condition and allowing for specification by proper choices of integrands. Every such moment condition corresponds with some particular $\psi-$weak topology, and as a basic observation, all the considered risk functionals are continuous w.r.t. the related $\psi-$weak topologies. We shall then identify sufficient growth conditions to integrands of risk functionals implying along with the continuity of the risk functionals the continuity of resulting objective function with respect to some particular $\psi-$weak topology induced by the growth condition. In general, $\psi-$weak topologies are finer than the topology of weak convergence. However, we may specify exactly those subsets, where they coincide. Hence in the last step we may point out the domains of stability for stochastic programming involving the considered general objective functions.

We shall apply the technical results to 2-stage mean-risk models unifying previous work. Moreover, this top-down approach foremostly yields verifiable continuity conditions for broader classes of risk functionals than before. For stochastic programs this enables extension of the continuity, and thus stability, analysis to more comprehensive classes of models.

The paper is organized as follows. Our main results on continuity of functions built upon general risk functionals are developed and formulated in section 2.
In section 3 we turn our attention to two-stage mean risk models. Here, the previous analysis yields the continuity of the objective function if the mentioned growth condition is fulfilled by the optimal value function of the recourse problem. After providing some preliminaries, sufficient conditions for a comprehensive mixed integer convex recourse model are derived. For specifications of the latter to quadratic and linear (mixed-integer) recourse, stronger results are obtained under weaker assumptions. Section 3 is concluded by implications towards stability of minimal values and sets of minimizers.
The essential notion of $\psi-$weak topologies will be introduced in Section 4. Their relationship with the topology of weak convergence will also be discussed there. Then we shall be ready to proof the main results in Section 5.

2. Main result. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space, i.e. it supports some random variable U which is uniformly distributed on $]0,1[$. Furthermore, let for $p \in [1,\infty[$ denote by $L^p(\Omega,\mathcal{F},\mathbb{P})$ the standard L^p-space w.r.t. $(\Omega,\mathcal{F},\mathbb{P})$, and let $\rho : L^p(\Omega,\mathcal{F},\mathbb{P}) \to \mathbb{R}$ be a convex mapping which is nondecreasing w.r.t. the $\mathbb{P}$-almost sure partial order. In addition we shall assume ρ to be law-invariant, i.e. $\rho(X) = \rho(Y)$ whenever X and Y have the same law under $\mathbb{P}$. Therefore, using notation $\mathcal{M}_1(\mathbb{R})$ for the set of all Borel-probability measures on $\mathbb{R}$, ρ induces a real-valued mapping $\mathcal{R}_\rho$ on

$$\mathcal{M}_1^{|\cdot|^p}(\mathbb{R}) := \{\mu \in \mathcal{P}(\mathbb{R}) \mid \int_{\mathbb{R}} |t|^p \mathrm{d}\mu(t) < \infty\}$$

defined by

$$\mathcal{R}_\rho(\mu) := \rho(F_\mu^{\leftarrow}(U)),$$

where $F_\mu^{\leftarrow}$ stands for the left-continuous quantile function of the distribution function F_μ of μ.

We shall consider stochastic minimization problems with goal functions of the form

$$Q(x,\nu) := \mathcal{R}_\rho\left((\delta_x \otimes \nu) \circ f^{-1}\right), \tag{2.1}$$

where $\delta_x \otimes \nu$ denotes the product probability measure of the Dirac measure at $x \in \mathbb{R}^{m_1}$ and some Borel probability measure ν on $\mathbb{R}^s$, whereas $(\delta_x \otimes \nu) \circ f^{-1}$ stands for the image measure of $\delta_x \otimes \nu$ under some fixed Borel-measurable mapping $f : \mathbb{R}^{m_1} \times \mathbb{R}^s \to \mathbb{R}$ which should belong to $\mathcal{M}_1^{|\cdot|^p}(\mathbb{R})$. Endowing $\mathbb{R}^s$ with the euclidean norm $\|\cdot\|_{s,2}$, we shall impose the following growth condition on the mapping f.

> There is some locally bounded mapping $\eta : \mathbb{R}^{m_1} \to]0,\infty[$ and a constant $\gamma > 0$ such that
>
> $$|f(x,z)| \leq \eta(x)(\|z\|_{s,2}^\gamma + 1) \quad \text{for all } (x,z) \in \mathbb{R}^{m_1} \times \mathbb{R}^s. \tag{2.2}$$

We want to analyze stability of optimization problems with such a kind of goal functions so that we are interested in conditions to guarantee weak continuity of $Q(x,\cdot)$ for $x \in \mathbb{R}^{m_1}$. Let us first fix a proper domain of $Q(x,\cdot)$.

For that purpose let $\mathcal{M}_1(\mathbb{R}^s)$ be the set of Borel-probability measures on $\mathbb{R}^s$, and define

$$\mathcal{M}_1^{\|\cdot\|_{s,2}^{\gamma p}}(\mathbb{R}^s) := \{\nu \in \mathcal{M}_1(\mathbb{R}^s) \mid \int_{\mathbb{R}^s} \|z\|_{s,2}^{\gamma p} \mathrm{d}\nu(z) < \infty\}.$$

It will turn out that under the growth condition $Q(x,\nu)$ is well-defined for $x \in \mathbb{R}^{m_1}$ and $\nu \in \mathcal{M}_1^{\|\cdot\|_{s,2}^{\gamma p}}(\mathbb{R}^s)$.

Lemma 2.1. *If (2.2) is satisfied, then* $\int_{\mathbb{R}^s} |f(x,z)|^p \mathrm{d}\nu(z) < \infty$ *holds for every* $x \in \mathbb{R}^{m_1}$ *and any* $\nu \in \mathcal{M}_1^{\|\cdot\|_{s,2}^{\gamma p}}(\mathbb{R}^s)$.

Proof. By (2.2) we may conclude for every $x \in \mathbb{R}^{m_1}$ and any $z \in \mathbb{R}^s$

$$|f(x,z)|^p \le \eta(x)^p(\|z\|_{s,2}^{\gamma} + 1)^p \le \eta(x)^p 2^p \max\{\|z\|_{s,2}^{p\gamma}, 1\} \le \eta(x)^p 2^p(\|z\|_{s,2}^{p\gamma} + 1).$$

Then the statement of Lemma 2.1 follows immediately. □

As it will be shown later on, we may achieve to show under the growth condition (2.2) the continuity of the mapping

$$Q : \mathbb{R}^{m_1} \times \mathcal{M}_1^{\|\cdot\|_{s,2}^{\gamma p}}(\mathbb{R}^s),\ (x,\nu) \mapsto \mathcal{R}_\rho\left((\delta_x \otimes \nu) \circ f^{-1}\right)$$

w.r.t. the product topology of the standard topology on $\mathbb{R}^{m_1}$ and some particular topology on $\mathcal{M}_1^{\|\cdot\|_{s,2}^{\gamma p}}(\mathbb{R}^s)$ which is finer than the ordinary topology of weak convergence. The key of our argumentation will be to point out those subsets $\mathcal{M}$ of $\mathcal{M}_1^{\|\cdot\|_{s,2}^{\gamma p}}(\mathbb{R}^s)$ where these topologies coincide. It will turn out (see Section 4 and in particular Proposition 4.2) that exactly the locally uniformly $\|\cdot\|_{s,2}^{\gamma p}$−integrating subsets $\mathcal{M}$ satisfy this property. Here a subset $\mathcal{M} \subseteq \mathcal{M}_1^{\|\cdot\|_{s,2}^{\gamma p}}(\mathbb{R}^s)$ will be called *locally uniformly* $\|\cdot\|_{s,2}^{\gamma p}$*−integrating* if for any $\nu \in \mathcal{M}$ there exists some open neighborhood $\mathcal{N}$ of ν w.r.t. the topology of weak convergence such that

$$\lim_{a\to\infty} \sup_{\mu \in \mathcal{N}\cap\mathcal{M}} \int_{\mathbb{R}^s} \|z\|_{s,2}^{\gamma p} \cdot \mathbb{1}_{]a,\infty[}(\|z\|_{s,2}^{\gamma p})\ \mu(dz) = 0.$$

In [20] several equivalent characterizations for locally uniformly $\|\cdot\|_{s,2}^{\gamma p}$−integrating sets of Borel-probability measures have been established, providing also a large amount of examples.

Now we are ready to formulate our main result.

Theorem 2.2. *Let* $\mathcal{M} \subseteq \mathcal{M}_1^{\|\cdot\|_{s,2}^{\gamma p}}(\mathbb{R}^s)$ *be a locally uniformly* $\|\cdot\|_{s,2}^{\gamma p}$*-integrating subset, and let* D_f *denote the set of discontinuity points of* f*. If* $x \in \mathbb{R}^{m_1}$ *and* $\nu \in \mathcal{M}$ *satisfy* $\delta_x \otimes \nu(D_f) = 0$*, then under the growth condition (2.2) the mapping* $Q|_{\mathbb{R}^{m_1}\times\mathcal{M}}$ *is continuous at* (x,ν) *with respect to the product topology of the standard topology on* $\mathbb{R}^{m_1}$ *and the relative topology of weak convergence on* $\mathcal{M}$.

Theorem 2.2 has the following specialization.

Corollary 2.3. *Let* $\mathcal{M} \subseteq \mathcal{M}_1^{\|\cdot\|_{s,2}^{\gamma p}}(\mathbb{R}^s)$ *be a locally uniformly* $\|\cdot\|_{s,2}^{\gamma p}$*-integrating subset, and let* D_f *denote the set of discontinuity points of* f*. If* $x \in \mathbb{R}^{m_1}$ *such that* $\{z \in \mathbb{R}^s \mid (x,z) \in D_f\}$ *has Lebesgue measure* 0*, and if* $\nu \in \mathcal{M}$ *is absolutely continuous w.r.t. the Lebesgue-Borel measure on* $\mathbb{R}^s$*, then under the growth condition (2.2) the mapping* $Q|_{\mathbb{R}^{m_1}\times\mathcal{M}}$ *is continuous at* (x,ν) *with respect to the product topology of the standard topology on* $\mathbb{R}^{m_1}$ *and the relative topology of weak convergence on* $\mathcal{M}$.

Proof. We have $\delta_x \otimes \nu(D_f) = 0$, so that Theorem 2.2 is applicable. □

3. Application to two-stage mean-risk models.

3.1. Mean-risk models. Consider the parametric optimization problem

$$\min_{x,y}\{c(x)+q(y) \mid x \in X, y \in C(x,z)\} \tag{3.1}$$

where $X \subseteq \mathbb{R}^{m_1}$ is nonempty, $c : \mathbb{R}^{m_1} \to \mathbb{R}$, $q : \mathbb{R}^{m_2} \to \mathbb{R}$ and $C : \mathbb{R}^{m_1} \times \mathbb{R}^s \to 2^{\mathbb{R}^{m_2}}$ are mappings and $z \in \mathbb{R}^s$ enters as a parameter. (3.1) gives rise to a two-stage random optimization problem if we assume $z = z(\omega)$ to be a s-dimensional random vector on some probability space and impose the information constraint that x has to be chosen without knowing the realization of $z(\omega)$. Furthermore, we assume the randomness to be purely exogenous, i.e. the distribution of z to not depend on x. After the observation of $z(\omega)$, the decision on $y = y(x,\omega)$ is taken best possible, namely as an optimal solution to the so called recourse problem

$$\min_{y}\{q(y) \mid y \in C(x,z(\omega))\}. \tag{3.2}$$

This problem induces a mapping $f : \mathbb{R}^{m_1} \times \mathbb{R}^s \to \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty,\infty\}$ defined by $f(x,z) := c(x) + \inf_y \{q(y) \mid y \in C(x,z)\}$. For every fixed x, $f(x,z(\omega))$ can be seen as an extend-real-valued random variable and a mean risk model is obtained by applying a weighted sum of the expectation and quantification of risk $\mathcal{R}$ to these random variables. Let

$$\rho(\cdot) = (\lambda\mathbb{E} + \lambda'\mathcal{R})(\cdot)$$

be such a weighted sum fitting in the framework of section 2, i.e. assume ρ to be defined on the regular L^p-space w.r.t. an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$, real valued, convex, law-invariant and nondecreasing w.r.t. the $\mathbb{P}$-almost sure partial order.

Under appropriate assumptions on the finiteness of the moments of $f(x,z(\omega))$, we obtain the optimization problem

$$\min_{x}\{Q(x,\mu) \mid x \in X\}, \tag{3.3}$$

where $\mu \in \mathcal{M}_1(\mathbb{R}^s)$ denotes the Borel-probability measure induced by the random variable $z(\cdot)$ and the objective function $Q(x,\mu) = \mathcal{R}_\rho((\delta_x \otimes \mu) \circ f^{-1})$ is exactly as in (2.1). Hence, if (3.2) is such that the growth condition (2.2) is fulfilled for the resulting function f, theorem 2.2 can be applied to derive the continuity of the restriction of Q w.r.t. the product topology of the standard topology on $\mathbb{R}^{m_1}$ and the relative topology of weak convergence on a suitable subset of $\mathcal{M}_1(\mathbb{R}^s)$. In the subsequent subsections, we identify classes of recourse models that yield the growth condition to be fulfilled.

3.2. Preliminaries. First we show that assuming c to be locally bounded allows us to restrict ourselves to the case where $c \equiv 0$.

LEMMA 3.1. *Let c be locally bounded, then growth condition* (2.2) *is fulfilled for f with constant γ if and only if it is fulfilled for $f - c$ with the same constant.*

Proof. "⇒": $|f(x,z) - c(x)| \le |f(x,z)| + |c(x)| \le \eta(x)(\|z\|_{s,2}^\gamma + 1)\ \forall (x,z) \in \mathbb{R}^{m_1} \times \mathbb{R}^s$, where $\eta(x) := \max\{\eta_f(x) + |c(x)|, 1\}$ is locally bounded and η_f denotes the locally bounded mapping from (2.2) for f.

"$\Leftarrow$": $|f(x,z)| \leq |f(x,z) - c(x)| + |c(x)| \leq \eta(x)(\|z\|_{s,2}^{\gamma} + 1)$ $\forall (x,z) \in \mathbb{R}^{m_1} \times \mathbb{R}^s$, where $\eta(x) := \max\{\eta_{f-c}(x) + |c(x)|, 1\}$ is locally bounded and η_{f-c} denotes the locally bounded mapping from (2.2) for $f - c$. □

Throughout the subsequent analysis we assume $C(x,z)$ to be given by integrality constraints and a finite system of inequalities where (x,z) only influences the right-hand side:

$$C(x,z) = \{y \in \mathbb{R}^{m_2^{\mathbb{R}}} \times \mathbb{Z}^{m_2^{\mathbb{Z}}} \mid g(y) \leq h(x,z)\}, \tag{3.4}$$

where $m_2^{\mathbb{R}} + m_2^{\mathbb{Z}} = m_2$, $g : \mathbb{R}^{m_2} \to \mathbb{R}^k$ and $h : \mathbb{R}^{m_1} \times \mathbb{R}^s \to \mathbb{R}^k$. Consider the mappings $C' : \mathbb{R}^k \to 2^{\mathbb{R}^{m_2}}$, $C'(t) := \{y \in \mathbb{R}^{m_2^{\mathbb{R}}} \times \mathbb{Z}^{m_2^{\mathbb{Z}}} \mid g(y) \leq t\}$ and $\varphi(t) : \mathbb{R}^k \to \bar{\mathbb{R}}$, $\varphi(t) := \inf\{q(y) \mid y \in C'(t)\}$.

Lemma 3.2. *Assume that c is locally bounded and there exist constants $\alpha, \beta, \kappa > 0$ and a locally bounded mapping $\eta' : \mathbb{R}^{m_1} \to]0,\infty[$ such that*

$$\|h(x,z)\|_{k,2} \leq \eta'(x)(\|z\|_{s,2}^{\alpha} + 1) \quad \textit{for all } (x,z) \in \mathbb{R}^{m_1} \times \mathbb{R}^s$$

and

$$|\varphi(t)| \leq \kappa(\|t\|_{k,2}^{\beta} + 1) \quad \textit{for all } t \in \mathbb{R}^k.$$

Then the growth condition 2.2 is fulfilled for f with constant $\gamma = \alpha * \beta$.

Proof. By lemma 3.1 we may assume $c \equiv 0$ without loss of generality. We have $|f(x,z)| = |\varphi(h(x,z))| \leq \kappa(\|h(x,z)\|_{k,2}^{\beta} + 1) \leq \kappa\eta'(x)^{\beta}((\|z\|_{s,2}^{\alpha} + 1)^{\beta} + 1)$ and by $(\|z\|_{s,2}^{\alpha} + 1)^{\beta} \leq 2^{\beta}(\|z\|_{s,2}^{\alpha*\beta} + 1)\}$ we obtain $|f(x,z)| \leq \eta(x)(\|z\|_{s,2}^{\alpha*\beta} + 1)$, where $\eta(x) := 2^{\beta+1}\kappa\eta'(x)^{\beta}$ is locally bounded. □

Remark 1. *If h is Lipschitz continuous, the assumption of the previous lemma is fulfilled with $\alpha = 1$.*

Lemma 3.3. *If $\mathrm{dom}|\varphi| \neq \emptyset$ and there exist constants $\beta, L > 0$ such that $|\varphi(t_1) - \varphi(t_2)| \leq L(\|t_1 - t_2\|_{k,2}^{\beta} + 1)$ for all $t_1, t_2 \in \mathbb{R}^k$, then there exists a constant $\kappa > 0$ such that $|\varphi(t)| \leq \kappa(\|t\|_{k,2}^{\beta} + 1)$ for all $t \in \mathbb{R}^k$.*

Proof. Fix $t_0 \in \mathrm{dom}|\varphi|$. By $|\varphi(0)| \leq |\varphi(0) - \varphi(t_0)| + |\varphi(t_0)| \leq L(\|t_0\|_{k,2}^{\beta} + 1) + |\varphi(t_0)|$ we conclude $0 \in \mathrm{dom}|\varphi|$. Hence, $|\varphi(t)| \leq |\varphi(t) - \varphi(0)| + |\varphi(0)| \leq \kappa(\|t\|_{k,2}^{\beta} + 1)$ for all $t \in \mathbb{R}^k$ and $\kappa := L + |\varphi(0)| \in]0,\infty[$. □

3.3. Mixed-integer convex recourse. In this subsection we consider recourse problems with convex objective function q and a feasible set $C(x,z)$ as in (3.4) where $g = (g_1, \ldots, g_k)^{\top}$ is such that g_i is convex and epi g_i is closed for every $i = 1, \ldots, k$. Using the notation from the previous subsection, lemmas 3.1 and 3.2 allow to focus our analysis on φ. Let $\bar{C}(t) := \{y \in \mathbb{R}^{m_2} \mid g(y) \leq t\}$ be the set we obtain by deleting the integrality constraints in $C'(t)$. The following assumptions ensure that φ is finite (see lemma 3.6):
(C1) $\bar{C}(0)$ is bounded.
(C2) $C'(t) \neq \emptyset$ for all $t \in \mathbb{R}^k$.

We will work with the following results:

Lemma 3.4. *Let $v : \mathbb{R}^{m_2} \to \mathbb{R}$ be convex. Then for every $r > 0$, v is Lipschitz continuous on $B_r^{m_2,2}(0) := \{y \in \mathbb{R}^{m_2} \mid \|y\|_{m_2,2} \leq r\}$ with constant $L(r) = \frac{2}{r}(|\max_{y\in\{2r,-2r\}^{m_2}} v(y)| + 2|v(0)|)$.*

Proof. See Lemma A and Theorem A in [27]. □

Theorem 3.5 (Corollary 5 from [3]). *Assume (C1) and (C2) and set $\Theta(y,t) := (\max\{g_1(y) - t_1, 0\}, \ldots, \max\{g_k(y) - t_k, 0\})^{\mathrm{T}} \in \mathbb{R}^k$. Then for every $R > 0$, there exists a constant*

$$K(R) = \sup_{t\in B_R^{k,2}(0),\ y\notin\bar{C}(t)} \frac{d_{\bar{C}(t)}(y)}{\|\Theta(y,t)\|_{k,\infty}} < \infty,$$

such that $d_\infty(\bar{C}(t_1), \bar{C}(t_2)) \leq K(R)\|t_1 - t_2\|_{k,2}$ for all $t_1, t_2 \in B_R^{k,2}(0)$.

Lemma 3.6. *Assume (C1) and (C2). Then φ is finite on $\mathbb{R}^k$ and the infimum is attained.*

Proof. Fix $t \in \mathbb{R}^k$. With $C'_{y^{\mathbb{Z}}}(t) := \{(y^{\mathbb{R}}, y^{\mathbb{Z}}) \in C'(t)\}$ we have $C'(t) = \bigcup_{y^{\mathbb{Z}}\in\mathbb{Z}^{m_2^{\mathbb{Z}}}} C'_{y^{\mathbb{Z}}}(t)$. Because of (C1), theorem 3.5 implies that $C'(t) \subseteq \bar{C}(t)$ is bounded. Consequently, $Z(t) := \{y^{\mathbb{Z}} \in \mathbb{Z}^{m_2^{\mathbb{Z}}} \mid C'_{y^{\mathbb{Z}}}(t) \neq \emptyset\}$ is finite. Furthermore, (C2) yields that $Z(t)$ is nonempty. By $C'_{y^{\mathbb{Z}}}(t) \subseteq C'(t)$ and lemma 3.4, q is continuous on each of the finitely many compact sets $C'_{y^{\mathbb{Z}}}(t)$, $y^{\mathbb{Z}} \in Z(t)$. Hence, $\varphi(t) = \min_{y^{\mathbb{Z}}\in Z(t)} \min_{y\in C'_{y^{\mathbb{Z}}}(t)} q(y)$ is finite. □

In view of the growth condition, we need to bound the growth of the constants $L(r)$ and $B(R)$:
(C3) $\exists \beta_1, L_1 > 0\ \forall r > 0 : L(r) \leq L_1 r^{\beta_1}$
(C4) $\exists \beta_2, L_2 > 0\ \forall R > 0 : K(R) \leq L_2 R^{\beta_2}$

Remark 2. *Assumption (C3) and the convexity of q are not needed if q is Lipschitz continuous on $\mathbb{R}^{m_2}$.*

Lemma 3.7. *Assume (C1)-(C4), then there exists a constant $\kappa \geq 0$ such that*

$$|\varphi(t)| \leq \kappa(\|t\|_{k,2}^{\beta_1+\beta_2+1} + 1),$$

for all $t \in \mathbb{R}^k$.

Proof. By Lemma 3.6, there exist $y \in C'(t)$, $y_0 \in C'(0)$ such that $\varphi(t) = q(y)$ and $\varphi(0) = q(y_0)$. Set $R := \|t\|_{k,2}$. We have

$$|\varphi(t)| = |q(y)| \leq |q(y) - q(y_0)| + |q(y_0)| \leq L(R)\|y - y_0\|_{k,2} + |q(y_0)| \tag{3.5}$$

and theorem 3.5 yields

$$\|y - y_0\|_{k,2} \leq d_\infty(\bar{C}(y), \bar{C}(0)) + \operatorname{diam}(\bar{C}(0)) \leq K(R)\|t\|_{k,2} + \operatorname{diam}(\bar{C}(0)). \tag{3.6}$$

Combining (3.5), (3.6), (C3) and (C4) yields $|\varphi(t)| \leq \kappa(\|t\|^{\beta_1+\beta_2+1} + 1)$ for $\kappa = 5L_1(L_2 + 2\text{diam}(\bar{C}(0))) + |q(y_0)|$. □

THEOREM 3.8. *Assume (C1)-(C4), that c is locally bounded and that there exist a constant $\alpha > 0$ and a locally bounded mapping $\eta' : \mathbb{R}^{m_1} \to]0, \infty[$ such that*

$$\|h(x,z)\|_{k,2} \leq \eta'(x)(\|z\|_{s,2}^{\alpha} + 1) \quad \textit{for all } (x,z) \in \mathbb{R}^{m_1} \times \mathbb{R}^s.$$

Then the growth condition 2.2 is fulfilled for f with constant $\gamma = \alpha(\beta_1 + \beta_2 + 1)$.

Proof. Combine lemmas 3.2 and 3.7. □

3.4. Quadratic integer recourse. In this subsection, we set $q(y) := d^\top y + y^\top Dy$ and $C(x,z) := \{y \in \mathbb{Z}^{m_2} \mid Ay \leq h(x,z)\}$, where $d \in \mathbb{R}^{m_2}$, $D = \text{diag}(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^{m_2 \times m_2}$ with $\lambda_1, \ldots, \lambda_m > 0$ and $A \in \mathbb{Z}^{k \times m_2}$. We use the notation $C'(t) := \{y \in \mathbb{Z}^{m_2} \mid Ay \leq t\}$ and work with the following assumptions:
(Q1) $\text{rank}A = m_2$.
(Q2) $C'(t) \neq \emptyset \; \forall t \in \mathbb{R}^k$.
(Q3) $\min\{d^\top y + y^\top Dy \mid Ay \leq t\}$ exists for all $t \in \mathbb{R}^k$.

Let $\Delta : \mathbb{R}^k \to \mathbb{R}$ be given by

$$\Delta(t) := \max\{|\det B| \mid \text{B is a square sub-matrix of } \begin{pmatrix} A & 0 & t \\ -2D & A^\top & d \end{pmatrix}\}.$$

LEMMA 3.9. *There exists a constant $\tau > 0$ such that $\Delta(x,z) \leq \tau(\|t\|_{k,2} + 1)$ for all $t \in \mathbb{R}^k$.*

Proof. Let B an arbitrary square sub-matrix of $\begin{pmatrix} A & 0 & t \\ -2D & A^\top & d \end{pmatrix}$. If B contains no entries from t, we have $\det(B) \leq \Delta(0)$. Otherwise, Laplace expansion along the column in question yields $\det(B) \leq \Delta(0)(k\|t\|_{k,\infty} + m_2\|d\|_{m_2,\infty})$.
Hence, $\Delta(t) \leq \tau(\|t\|_{k,2} + 1)$, where $\tau := (k + m_2\|d\|_{m_2,\infty})\Delta(0)$. □

LEMMA 3.10. *Assume (Q1)-(Q3), then $\varphi(t) := \inf_y\{d^\top + y^\top Dy \mid y \in C'(t)\}$ is finite, the infimum is attained and there exists a constant $\kappa \geq 0$ such that*

$$|\varphi(t)| \leq \kappa(\|t\|_{k,2}^2 + 1)$$

holds true for all $t \in \mathbb{R}^k$.

Proof. Fix $t \in \mathbb{R}^k$. By Theorem 6 in [14], φ is finite, the infimum is attained and there exists a constant $\sigma > 0$ depending only on A, d and D, such that

$$|\varphi(t)| \leq |\varphi(t) - \varphi(0)| + |\varphi(0)| \leq \sigma(\Delta(t)\|t\|_{k,2} + 1)$$

and Lemma 3.9 implies $|\varphi(t)| \leq \kappa(\|t\|_{k,2}^2 + 1)$, where $\kappa := \sigma + \tau + 1$. □

THEOREM 3.11. *Assume (Q1)-(Q3), that c is locally bounded and that there exist a constant $\alpha > 0$ and a locally bounded mapping $\eta' : \mathbb{R}^{m_1} \to]0, \infty[$ such that*

$$\|h(x,z)\|_{k,2} \leq \eta'(x)(\|z\|_{s,2}^{\alpha} + 1) \quad \textit{for all } (x,z) \in \mathbb{R}^{m_1} \times \mathbb{R}^s.$$

Then the growth condition 2.2 is fulfilled for f with constant $\gamma = 2\alpha$.

Proof. Combine lemmas 3.2 and 3.10. □

LEMMA 3.12. *Assume (Q1)-(Q3), that c is continuous and that h is such that $h^{-1}((\mathbb{R}^{m_1} \times \mathbb{R}^s) \setminus \mathcal{X})$ has Lebesgue measure zero, where $\mathcal{X} := (\mathbb{R} \setminus \mathbb{Z})^{m_1} \times (\mathbb{R} \setminus \mathbb{Z})^s$. Then the set D_f of discontinuities of f has Lebesgue measure zero.*

Proof. Let D_{f-c} denote the set of discontinuities of $f - c$. Because of $A \in \mathbb{Z}^{k \times m_2}$, C' and $f - c$ are locally constant around every $(x, z) \in h^{-1}(\mathcal{X})$. Consequently, $D_f = D_{f-c} \subseteq h^{-1}((\mathbb{R}^{m_1} \times \mathbb{R}^s) \setminus \mathcal{X})$. □

3.5. Mixed-integer linear recourse. In this subsection, we set $q(y) := d^\top y$ and $C(x, z) := \{y = (y^{\mathbb{R}}, y^{\mathbb{Z}}) \in \mathbb{R}^{m_2^{\mathbb{R}}} \times \mathbb{Z}^{m_2^{\mathbb{Z}}} \mid Wy = W^{\mathbb{R}} y^{\mathbb{R}} + W^{\mathbb{Z}} y^{\mathbb{Z}} = h(x, z), y \geq 0\}$, where $d \in \mathbb{R}^{m_2}$, $m_2^{\mathbb{R}} + m_2^{\mathbb{Z}} = m_2$, $W \in \mathbb{Q}^{k \times m_2}$. We will work with the following assumptions:

(A1) $W(\mathbb{R}_{\geq 0}^{m_2^{\mathbb{R}}} \times \mathbb{Z}_{\geq 0}^{m_2^{\mathbb{Z}}}) = \mathbb{R}^k$.

(A2) $\{u \in \mathbb{R}^k \mid W^\top u \leq d\} \neq \emptyset$.

Set $C'(t) := \{y = (y^{\mathbb{R}}, y^{\mathbb{Z}}) \in \mathbb{R}^{m_2^{\mathbb{R}}} \times \mathbb{Z}^{m_2^{\mathbb{Z}}} \mid Wy = W^{\mathbb{R}} y^{\mathbb{R}} + W^{\mathbb{Z}} y^{\mathbb{Z}} = t, y \geq 0\}$ and $\varphi(t) := \inf_y \{d^\top y \mid y \in C'(t)\}$.

LEMMA 3.13. *Assume (A1) and (A2), then $\varphi(t) := \inf_y \{d^\top y \mid y \in C'(t)\}$ is finite, the infimum is attained and there are constants $\beta, \kappa > 0$ such that $|\varphi(t)| \leq \kappa(\|t\|_{k,2} + 1)$ for all $t \in \mathbb{R}^k$.*

Proof. By [10], φ is finite and the infimum is attained. Furthermore, there exist constants $L, \beta > 0$ such that $|\varphi(t_1) - \varphi(t_2)| \leq L(\|t_1 - t_2\|_{k,2}^{\beta} + 1)$ for all $t_1, t_2 \in \mathbb{R}^k$. Hence, 3.3 is applicable. □

THEOREM 3.14. *Assume (A1), (A2), that c is locally bounded and that there exist a constant $\alpha > 0$ and a locally bounded mapping $\eta' : \mathbb{R}^{m_1} \to]0, \infty[$ such that*

$$\|h(x, z)\|_{k,2} \leq \eta'(x)(\|z\|_{s,2}^{\alpha} + 1) \quad \textit{for all } (x, z) \in \mathbb{R}^{m_1} \times \mathbb{R}^s.$$

Then the growth condition 2.2 is fulfilled for f with constant $\gamma = \alpha * \beta$, where β denotes the constant from lemma 3.13.

Proof. Combine lemmas 3.2 and 3.13. □

LEMMA 3.15. *Assume (A1), (A2), that c is continuous and that h is such that $h^{-1}(\bigcup_{y^{\mathbb{Z}} \in \mathbb{Z}_{\geq 0}^{m_2^{\mathbb{Z}}}} \{W^{\mathbb{Z}} y^{\mathbb{Z}} + \operatorname{bd} W^{\mathbb{R}}(\mathbb{R}_{\geq 0}^{m_2^{\mathbb{R}}})\})$ has Lebesgue measure zero. Then the set D_f of discontinuities of f has Lebesgue measure zero.*

Proof. See [23].

3.6. Implications for stability. Let $\varphi : \mathcal{M}_s^{\gamma p} \to \bar{\mathbb{R}}$, $\varphi(\mu) := \inf\{Q(x, \mu) \mid x \in X\}$ denote the optimal value function of problem (3.3).

COROLLARY 1. *Let the assumptions of Lemma 3.12 or Lemma 3.15 be fulfilled. Then $\varphi|_{\mathcal{M}}$ is upper semicontinuous on $\mathcal{M}$ with respect the relative topology of weak* *convergence.*

Proof. Under the given assumptions, $Q|_{X\times\mathcal{M}}$ is continuous on $X \times \mathcal{M}$. Since the feasible set X is fixed, that yields the upper semicontinuity of $\varphi|_{\mathcal{M}}$ (see section 4.1 in [11]). □

COROLLARY 2. *Let the assumptions of Lemma 3.12 or Lemma 3.15 be fulfilled and assume that X is compact. Then $\varphi|_{\mathcal{M}}$ is continuous on $\mathcal{M}$ and the restricted optimal solution set mapping $\Phi|_{\mathcal{M}} : \mathcal{M} \to 2^{\mathbb{R}^n}$, $\Phi|_{\mathcal{M}}(\mu) = \operatorname{argmin}\{Q(x,\mu) \mid x \in X\}$ is upper semicontinuous on $\mathcal{M}$ with respect to the relative topology of weak convergence, i.e. for any $\mu_0 \in \mathcal{M}$ and any open set $\mathcal{O} \subseteq \mathbb{R}^n$ such that $\Phi|_{\mathcal{M}}(\mu_0) \subseteq \mathcal{O}$ there exists a neighborhood $\mathcal{N}$ of μ_0 with respect to the topology of weak convergence such that $\Phi|_{\mathcal{M}}(\mu) \subseteq \mathcal{O}$ for all $\mu \in \mathcal{N}$.*

Proof. See Proposition 1.1 in [34]. □

□

4. $\psi-$weak topology and the topology of weak convergence. Let $\psi : \mathbb{R}^d \to [0,\infty[$ be a continuous function such that $\psi \geq 1$ outside a compact set. Such a function will be referred to as a *gauge function*. Let $\mathcal{M}_1^{\psi}(\mathbb{R}^d)$ be the set of all Borel-probability measures μ on $\mathbb{R}^d$ satisfying $\int \psi \, d\mu < \infty$, and $C_{\psi}(\mathbb{R}^d)$ be the space of all continuous functions $g : \mathbb{R} \to \mathbb{R}$ for which $\sup_{y\in\mathbb{R}} |g(y)/(1+\psi(y))| < \infty$. The ψ-weak topology on $\mathcal{M}_1^{\psi}(\mathbb{R}^d)$ is defined to be the coarsest topology for which all mappings $\mu \mapsto \int g \, d\mu$, $g \in C_{\psi}(\mathbb{R}^d)$, are continuous; cf. Section A.6 in [13]. The $\psi-$weak topology may be characterized in the following ways.

LEMMA 4.1. *The $\psi-$weak topology is metrizable, and for every sequence $(\mu_n)_{n\in\mathbb{N}_0}$ in $\mathcal{M}_1^{\psi}(\mathbb{R}^d)$ the following statements are equivalent.*

(1) $\mu_n \to \mu_0$ w.r.t. the $\psi-$weak topology.

(2) $\mu_n \to \mu_0$ w.r.t. the topology of weak convergence, and $\int_{\mathbb{R}^d} \psi \, d\mu_n \to \int_{\mathbb{R}^d} \psi \, d\mu_0$.

(3) $\mu_n \to \mu_0$ w.r.t. the topology of weak convergence, and

$$\lim_{a\to\infty} \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} \psi \cdot \mathbb{1}_{]a,\infty[}(\psi) \, d\mu_n = 0.$$

Proof. From Theorem A.38 in [13] it is known that the the $\psi-$weak topology is metrizable. The equivalence of (1) and (2) has been shown in [17, Lemma 3.4]. Finally, the equivalence of (2) and (3) follows immediately from the convergence of moments theorem (cf. [39, Theorem 2.20]). □

We are interested in gauge functions of the form $\psi := \|\cdot\|_{d,2}^q$ for any $q > 0$, where $\|\cdot\|_{d,2}$ stands for the euclidean norm on $\mathbb{R}^d$. For $q \geq 1$, the $\|\cdot\|_{d,2}^q-$weak topology is generated by so called *Kantorovich metric* $d_{K,d,q}$ w.r.t. $\|\cdot\|_{d,2}^q$ defined by

$$d_{K,d,q}(\mu,\nu) := \inf\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} \|x-y\|_{d,2}^q \, \pi(dx,dy) \mid \pi \in \mathcal{M}_1(\mu,\nu)\right\},$$

where $\mathcal{M}_1(\mu,\nu)$ denotes the set of all Borel-probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with μ as the first $d-$dimensional marginal and ν as the second one (cf. [25, Theorem 6.3.1]). If $d = 1$, then the Kantorovich metric $d_{K,d,1}$ coincides with the so called *Wasserstein metric of order* q. Moreover, in this case the $|\cdot|^q-$weak topology may be also generated by the *Fortet-Mourier metric* d_{FMq} *of order* q defined by

$$d_{FM,q}(\mu,\nu) := \inf\left\{\int_{\mathbb{R}\times\mathbb{R}} |x-y|^q \, \pi(dx,dy) \mid \pi \in \mathcal{L}_1(\mu,\nu)\right\},$$

where $\mathcal{L}_1(\mu,\nu)$ denotes the set of all finite Borel measures on $\mathbb{R}\times\mathbb{R}$ satisfying

$$\pi(A\times\mathbb{R})-\pi(\mathbb{R}\times A)=\mu(A)-\nu(A)\quad\text{for every Borel-subset } A\subseteq\mathbb{R}$$

(see [25, Theorem 6.2.1]).

Next we want to identify those subsets $\mathcal{M}\subseteq\mathcal{M}_1^{\|\cdot\|_{d,2}^q}(\mathbb{R}^d)$ where the relative $\|\cdot\|_{d,2}^q-$weak topology coincides with the relative topology of weak convergence. Analogously to Section 2 we shall call a subset $\mathcal{M}\subseteq\mathcal{M}_1^{\|\cdot\|_{d,2}^q}(\mathbb{R}^d)$ to be *locally uniformly* $\|\cdot\|_{d,2}^q-$*integrating* if for any $\nu\in\mathcal{M}$ there exists some open neighbourhood $\mathcal{N}$ of ν w.r.t. the topology of weak convergence such that

$$\lim_{a\to\infty}\sup_{\mu\in\mathcal{N}\cap\mathcal{M}}\int_{\mathbb{R}^d}\|z\|_{s,2}^q\cdot\mathbb{1}_{]a,\infty[}(\|z\|_{s,2}^q)\ \mu(dz)=0.$$

Proposition 4.2. *For $q>0$ and $\mathcal{M}\subseteq\mathcal{M}_1^{\|\cdot\|_{d,2}^q}(\mathbb{R}^d)$ the following statements are equivalent.*

(1) On $\mathcal{M}$ the relative $\|\cdot\|_{d,2}^q-$weak topology coincides with the relative topology of weak convergence.

(2) $\mathcal{M}$ is locally uniformly $\|\cdot\|_{d,2}^q-$ integrating.

Proof.
Assuming statement (2) and drawing on Lemma 4.1, we obtain for any sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ and $\mu\in\mathcal{M}_1^{\|\cdot\|_{d,2}^q}(\mathbb{R}^d)$

$$\mu_n\to\mu_0\ \|\cdot\|_{d,2}^q-\text{weakly}\quad\Leftrightarrow\quad\mu_n\to\mu\text{ w.r.t. the topology of weak convergence.}$$

Since the $\|\cdot\|_{d,2}^q-$weak topology is metrizable, this shows implication (2) $\Rightarrow$ (1).

Concerning the implication (1) $\Rightarrow$ (2) let the statement (1) be true, and let us assume that statement (2) does not hold. Then we may find a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ which converges to some $\mu\in\mathcal{M}_1^{\|\cdot\|_{d,2}^q}(\mathbb{R}^d)$ w.r.t. to the topology of weak convergence such that

$$\lim_{a\to\infty}\sup_{n\in\mathbb{N}}\int_{\mathbb{R}^d}\|z\|_{s,2}^q\cdot\mathbb{1}_{]a,\infty[}(\|z\|_{s,2}^q)\ \mu_n(dz)>0.$$

Thus on the one hand $(\mu_n)_{n\in\mathbb{N}}$ does not converge to μ w.r.t. the $\|\cdot\|_{d,2}^q-$topology. On the other hand by statement (1), $\mu_n\to\mu$ $\|\cdot\|_{d,2}^q-$weakly which is a contradiction. Therefore the implication (1) $\Rightarrow$ (2) is valid which completes the proof. □

5. Proof of the main result. Consider the mapping

$$\Phi_f:\mathbb{R}^{m_1}\times\mathcal{M}_1^{\|\cdot\|_{s,2}^{\gamma p}}(\mathbb{R}^s)\to\mathcal{M}_1^{|\cdot|^p}(\mathbb{R}),\ (x,\nu)\mapsto(\delta_x\otimes\nu)\circ f^{-1}$$

which is well-defined under growth condition (2.2) due to Lemma 2.1. Let us equip $\mathcal{M}^{\|\cdot\|_{s,2}^{\gamma p}}(\mathbb{R}^s)$ and $\mathcal{M}^{|\cdot|^p}(\mathbb{R})$ respectively with the $\|\cdot\|_{s,2}^{\gamma p}-$weak topology $\tau_{s,\gamma p}$ and $|\cdot|^p-$weak topology $\tau_{1,p}$ as defined in Section 4. Furthermore, let $\tau_{\mathbb{R}^{m_1}}\otimes\tau_{s,\gamma p}$ denote the product topology of the standard topology on $\mathbb{R}^{m_1}$ and $\tau_{s,\gamma p}$. An important step in our argumenation is to investigate continuity of the mapping Φ_f w.r.t. $\tau_{\mathbb{R}^{m_1}}\otimes\tau_{s,\gamma p}$ and $\tau_{1,p}$.

Proposition 5.1. *Let D_f be the set of discontinuity points of f, and let the growth condition (2.2) be fulfilled. Furthermore let $x \in \mathbb{R}^{m_1}$ and $\nu \in \mathcal{M}_1^{\|\cdot\|_{s,2}^{\gamma p}}(\mathbb{R}^s)$ such that $\delta_x \otimes \nu(D_f) = 0$. Then Φ_f is continuous at (x, ν) w.r.t. $\tau_{\mathbb{R}^{m_1}} \otimes \tau_{s,\gamma p}$ and $\tau_{1,p}$.*

Proof. Since $\tau_{\mathbb{R}^{m_1}} \otimes \tau_{s,\gamma p}$ and $\tau_{1,p}$ are metrizable, it suffices to show that Φ_f is sequentially continuous at (x, ν) w.r.t. $\tau_{\mathbb{R}^{m_1}} \otimes \tau_{s,\gamma p}$ and $\tau_{1,p}$.

So let $(x_n)_{n\in\mathbb{N}}$ be any sequence in $\mathbb{R}^{m_1}$ converging to x, and let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_1^{\|\cdot\|_{s,2}^{\gamma p}}(\mathbb{R}^s)$ such $\nu_n \to \nu$ $\|\cdot\|_{s,2}^{\gamma p}-$weakly.
Then obviously
$\delta_{x_n} \otimes \nu_n \to \delta_x \otimes \nu$ with respect to the relative topology of weak convergence on $\mathbb{R}^{m_1} \times \mathbb{R}^s$. By assumption $\delta_x \otimes \nu(D_f) = 0$ holds so that we may conclude from
the continuous mapping theorem (see e.g. [9, Theorem 5.1]) that $(\delta_{x_n} \otimes \nu_n) \circ f^{-1} \to (\delta_x \otimes \nu) \circ f^{-1}$ w.r.t. the relative topology of weak convergence on $\mathbb{R}$.

Furthermore, as in the proof of Lemma 2.1 we obtain

$$|f(x_n, z)|^p \leq \eta(x_n)^p 2^p (\|z\|_{s,2}^{p\gamma} + 1) \quad \text{for } n \in \mathbb{N},\ z \in \mathbb{R}^s.$$

Since the mapping η is locally bounded, we may assume without loss of generality that $C := \sup_{n\in\mathbb{N}} \eta(x_n) < \infty$. Thus by Fubini-Tonelli theorem

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^{m_1}\otimes\mathbb{R}^s} |f|^p \mathbb{1}_{]a,\infty[}(|f|^p) \mathrm{d}(\delta_{x_n} \otimes \nu_n) \\
&\leq 2^p C^p \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^{m_1}\times\mathbb{R}^s} (\|z\|^{\gamma p} + 1) \mathbb{1}_{]a/(2^p C^p),\infty[}(\|z\|^{\gamma p} + 1)\ (\delta_{x_n} \otimes \nu_n)(\mathrm{d}y, \mathrm{d}z) \\
&= 2^p C^p \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^s} (\|z\|^{\gamma p} + 1) \mathbb{1}_{]a/(2^p C^p),\infty[}(\|z\|^{\gamma p} + 1)\ \nu_n(\mathrm{d}z)
\end{aligned}$$

Since $\nu_n \to \nu$ $\|\cdot\|_{s,2}^{\gamma p}-$weakly, we may conclude from Lemma 4.1

$$\lim_{a\to\infty} \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^s} (\|z\|^{\gamma p} + 1) \mathbb{1}_{]a/(2^p C^p),\infty[}(\|z\|^{\gamma p} + 1)\ \nu_n(\mathrm{d}z) = 0.$$

This implies

$$\lim_{a\to\infty} \sup_{n\in\mathbb{N}} \int_{\mathbb{R}} |\cdot|^p \mathrm{d}(\delta_{x_n} \otimes \nu_n) \circ f^{-1} = \lim_{a\to\infty} \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^{m_1}\otimes\mathbb{R}^s} |f|^p \mathbb{1}_{]a,\infty[}(|f|^p) \mathrm{d}(\delta_{x_n} \otimes \nu_n) = 0,$$

so that by Lemma 4.1 again

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathbb{R}} |\cdot|^p \mathrm{d}(\delta_{x_n} \otimes \nu_n) \circ f^{-1} &= \lim_{n\to\infty} \int_{\mathbb{R}^{m_1}\times\mathbb{R}^s} |f|^p \mathrm{d}(\delta_{x_n} \otimes \nu_n) \\
&= \int_{\mathbb{R}^{m_1}\times\mathbb{R}^s} |f|^p \mathrm{d}(\delta_x \otimes \nu) = \int_{\mathbb{R}} |\cdot|^p \mathrm{d}(\delta_x \otimes \nu) \circ f^{-1}.
\end{aligned}$$

Therefore $(\delta_{x_m} \otimes \nu_m) \circ f^{-1} \to (\delta_x \otimes \nu) \circ f^{-1}$ w.r.t. the $\|\cdot\|_{s,2}^{\gamma p}-$weak topology. □

In the next step we want to show the continuity of $\mathcal{R}_\rho$ w.r.t. the $|\cdot|^p$-weak topology.

Theorem 5.2. *R_ρ is continuous with respect to the $|\cdot|^p$-weak topology.*

Proof. Let us first recall that $L^p(\Omega, \mathcal{F}, \mathbb{P})$, equipped with ordinary L^p-norm $\|\cdot\|_p$ and the $\mathbb{P}$-almost sure partial order, is a Banach lattice, i.e. $\|\cdot\|_p$ is complete satisfying

$\|X\|_p \leq \|Y\|_p$ whenever $|X| \leq |Y|$ $\mathbb{P}-$a.s. (see Theorem 13.5 in [1]). Furthermore ρ is assumed to be a convex mapping on the Banach lattice $L^p(\Omega, \mathcal{F}, \mathbb{P})$ which is nondecreasing w.r.t. the $\mathbb{P}$-almost sure partial order. Hence it is continuous w.r.t. $\|\cdot\|_p$ (cf.[8, Corollary 2]).

Since the $|\cdot|^p-$weak topology is metrizable it suffices to show sequential continuity for $\mathcal{R}_\rho$. So consider any sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_1^{|\cdot|^p}(\mathbb{R})$ which converges to some $\mu \in \mathcal{M}_1^{|\cdot|^p}(\mathbb{R})$ w.r.t. the $|\cdot|^p$-weak topology. Then by Theorem 3.5 in [18] we may find a sequence $(X_n)_{n\in\mathbb{N}}$ in $L^p(\Omega, \mathcal{F}, \mathbb{P})$ and some $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$ such that μ_n is the distribution of X_n for $n \in \mathbb{N}$, X has μ as its distribution, and $\int_\Omega |X_m - X|^p d\mathbb{P} \to 0$. So by law-invariance and norm-continuity of ρ we have

$$R_\rho(\mu_n) = \rho(X_n) \to \rho(X) = R_\rho(\mu).$$

This completes the proof. □

Combining Proposition 5.1 with Theorem 5.2, we obtain immediately the following criterion to guarantee continuity of the function

$$Q : \mathbb{R}^{m_1} \times \mathcal{M}_1^{\|\cdot\|_{s,2}^{\gamma p}}(\mathbb{R}^s),\ (x, \nu) \mapsto \mathcal{R}_\rho\left((\delta_x \otimes \nu) \circ f^{-1}\right)$$

w.r.t. the production topology of the standard topology on $\mathbb{R}^{m_1}$ and the $\|\cdot\|_{s,2}^{\gamma p}-$weak topology on $\mathcal{M}_1^{\|\cdot\|_{s,2}^{\gamma p}}(\mathbb{R}^s)$.

THEOREM 5.3. *Let D_f denote the sets of discontinuity points of f. If $x \in \mathbb{R}^{m_1}$ and $\nu \in \mathcal{M}$ satisfy $\delta_x \otimes \nu(D_f) = 0$, then under the growth condition (2.2) the mapping Q is continuous at (x, ν) with respect to the product topology of the standard topology on $\mathbb{R}^{m_1}$ and the $\|\cdot\|_{s,2}^{\gamma p}-$weak topology on $\mathcal{M}_1^{\|\cdot\|_{s,2}^{\gamma p}}(\mathbb{R}^s)$.*

Now we are ready to prove the main result.

Proof of Theorem 2.2:
Since $\mathcal{M}$ is assumed to be locally uniformly $\|\cdot\|_{s,2}^{\gamma p}-$integrating, the topology of weak convergence and the $\|\cdot\|_{s,2}^{\gamma p}$-weak topology coincide on $\mathcal{M}$ due to Proposition 4.2. Thus Theorem 2.2 follows immediately from Theorem 5.3. □

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