Least-Squares Spectral Collocation with the Overlapping Schwarz Method for the Incompressible Navier–Stokes Equations

by

Wilhelm Heinrichs
Universität Duisburg–Essen, Ingenieurmathematik
Universitätsstr. 3, D-45117 Essen, Germany
E-Mail: wheinric@ing-math.uni-essen.de

Abstract

A least-squares spectral collocation scheme is combined with the overlapping Schwarz method. The methods are succesfully applied to the incompressible Navier-Stokes equations. The collocation conditions and the interface conditions lead to an overdetermined system which can be efficiently solved by least-squares. The solution technique will only involve symmetric positive definite linear systems. The overlapping Schwarz method is used for the iterative solution. For parallel implementation the subproblems are solved in a checkerboard manner. Our approach is successfully applied to the lid-driven cavity flow problem. Only a few Schwarz iterations are necessary in each time step. Numerical simulations confirm the high accuracy of our spectral least-squares scheme.

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1 Introduction

Spectral methods (see, e.g., Canuto et al. [5], Gottlieb and Orszag [11], [22] or Deville et al. [7]) employ global polynomials for the numerical solution of differential equations. Hence they give very accurate approximations for smooth solutions with relatively few degrees of freedom. For analytical data exponential convergence can be achieved. For problems with non-smooth solutions (e.g., discontinuities or layers) the usual (global) continuous spectral approach yields very poor approximation results. For this purpose the original domain has to be decomposed into several subdomains where jumps at the interfaces are allowed. Gerritsma and Proot [10] showed the good performance of discontinuous least-squares spectral element methods. In [15] we extended the above approach to one-dimensional singular perturbation problems where the least-squares collocation schemes lead to a stabilization. The collocation conditions together with the interface conditions lead to an overdetermined system which can be approximately solved by least-squares. We obtain symmetric and positive definite systems which can be efficiently solved by (banded) Cholesky or (preconditioned) conjugate gradient methods.

Here we consider the Stokes and Navier-Stokes equations. From the finite element case it is already known that the least-squares formulation of the Stokes [6, 19, 20] and Navier-Stokes equations [18, 21] leads to symmetric and positive definite algebraic systems which circumvent the $Babuška - Brezzi$ stability condition. For spectral methods it is known that if the velocity and the pressure are approximated by polynomials of the same degree eight spurious modes are introduced which lead to an instable system (see Bernardi, Canuto and Maday [1]). A well-known compatible approximating velocity-pressure pair is the so-called $\mathbb{P}_N \times \mathbb{P}_{N-2}$ formulation of Bernardi, Maday [2] and Rønquist [26]. Heinrichs [12, 14] employed this technique for the splitting of the Stokes equations. Here the velocity components are approximated by polynomials in $\mathbb{P}_N$ and the pressure by two degrees lower order polynomials in $\mathbb{P}_{N-2}$. The resulting discrete system constitutes a saddle point problem which is difficult to solve numerically. Least-squares techniques offer theoretical and numerical advantages over the classical methods. Spectral least-squares methods for the Stokes and Navier-Stokes equations were first introduced by Gerritsma et al. [9, 23, 24, 25]. Heinrichs [16] investigated least-squares collocation schemes for the Navier-Stokes equations. Here we extend these methods to an overlapping spectral element decomposition of the domain. The domain decomposition problem is iteratively solved by means of the Schwarz method. At the begin of the time integration we use some nested Schwarz method. For the efficient parallel implementation we solve the subproblems in a checkerboard manner. The methods are successfully applied to the lid-driven cavity flow problem. It becomes obvious that in each time step only a few Schwarz steps are necessary. Summarizing our approach has the following advantages:

- equal order interpolation polynomials can be employed
- improved stability properties for singular perturbation problems [8, 13, 15] and the Navier-Stokes equations [9, 16, 23, 24, 25]
- good performance in combination with the overlapping Schwarz method
- direct or iterative solvers for positive definite systems (e.g., Cholesky or conjugate gradient methods) can be used
- implementation and parallelization is straightforward.

The paper is organized as follows. In Section 2, the first-order formulation of the Navier-Stokes equations is introduced. Then we describe the spectral least-squares spectral collocation scheme (section 3) and the overlapping Schwarz method (section 4). In section 5 we present numerical results for a smooth example and the (regularized and lid-) driven cavity flow problem. Finally a conclusion is presented.

2 The Navier–Stokes Equations

In order to apply least-squares the Navier-Stokes problem is transformed into an equivalent first-order system of partial differential equations. This is accomplished by introducing the vorticity $\omega = \nabla \times \mathbf{u}$ as an auxiliary variable. By using the identity

$$\nabla \times \nabla \times \mathbf{u} = -\Delta \mathbf{u} + \nabla(\nabla \cdot \mathbf{u})$$

and by using the incompressibility constraint $\nabla \cdot \mathbf{u} = 0$ we obtain

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p + \nu \nabla \times \omega &= \mathbf{f} \;\; in\ \Omega && (1) \\ \nabla \cdot \mathbf{u} &= 0 \;\; in\ \Omega && (2) \\ \omega - \nabla \times \mathbf{u} &= 0 \;\; in\ \Omega && (3) \end{aligned}$$

where $\mathbf{u}^T = [u, v]$ denotes the velocity vector, p the pressure, $\mathbf{f}^T = [f_u, f_v]$ the forcing term and ν the kinematic viscosity. Here it is assumed that the density equals unity. As proposed in [25] we apply a θ-integration scheme in time combined with the Picard linearization to the momentum equation of the unsteady Navier-Stokes equations. The subscript "0" corresponds to the results obtained at a previous integration time step. Now the momentum equation reads as follows:

$$\begin{aligned} &\frac{\mathbf{u} - \mathbf{u}_0}{\Delta t} + \theta \left(\mathbf{u}_0 \cdot \nabla \mathbf{u} + \nabla p + \nu \nabla \times \omega - \mathbf{f} \right) && (4) \\ &= (\theta - 1) \left(\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 + \nabla p_0 + \nu \nabla \times \omega_0 - \mathbf{f}_0 \right). && (5) \end{aligned}$$

By taking $\theta = 1$ the time integration reduces to backward Euler which is only first order accurate in time. The second order time integration of Crank-Nicolson can be obtained by setting $\theta = 1/2$. Since the Crank-Nicolson scheme has no damping one often takes $\theta = 1/2 + O(\Delta t)$. The temporal accuracy remains second order and adding the small factor Δt effectively damps the small waves in spectral element simulations. Hence in order to obtain time accurate solutions one should use $\theta = 1/2 + O(\Delta t)$. The $\theta-$scheme is unconditionally stable for $1/2 \leq \theta \leq 1$. Here we only consider stationary problems where it is recommended to use backward Euler ($\theta = 1$) with a large time step to obtain steady state solutions. Now the complete system for each time step can explicitly be written as

$$\begin{bmatrix} \frac{1}{\Delta t} + \theta u_0 \frac{\partial}{\partial x} + \theta v_0 \frac{\partial}{\partial y} & 0 & \theta \nu \frac{\partial}{\partial y} & \theta \frac{\partial}{\partial x} \\ 0 & \frac{1}{\Delta t} + \theta u_0 \frac{\partial}{\partial x} + \theta v_0 \frac{\partial}{\partial y} & -\theta \nu \frac{\partial}{\partial x} & \theta \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 1 & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ \omega \\ p \end{bmatrix} = \mathbf{F}$$

where

$$\mathbf{F} = \begin{bmatrix} f_u + \frac{u_0}{\Delta t} + (\theta - 1)\left\{u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} + \nu \frac{\partial \omega_0}{\partial y} + \frac{\partial p_0}{\partial x}\right\} \\ f_v + \frac{v_0}{\Delta t} + (\theta - 1)\left\{u_0 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial v_0}{\partial y} - \nu \frac{\partial \omega_0}{\partial x} + \frac{\partial p_0}{\partial y}\right\} \\ 0 \\ 0 \end{bmatrix}.$$

3 The Spectral Derivatives

Now in the spectral scheme all four functions u, v, ω, p are approximated by polynomials in $\mathbb{P}_N$. Furthermore we have to introduce the standard Chebyshev Gauss-Lobatto collocation nodes which are explicitly given by

$$(x_i, y_j) = \left(\cos\frac{i\pi}{N}, \cos\frac{j\pi}{N}\right),\ i, j = 0, \ldots, N.$$

In the following we write the spectral derivatives. First one has to introduce the transformation matrices from physical space to coefficient space. Since we employ a Chebyshev expansion we obtain the following matrix:

$$T \;=\; \left(\cos(k\frac{i\pi}{N})\right),\ i, k = 0, \ldots, N.$$

Further we need the differentation matrix in the Chebyshev coefficient space which is explicitly given by $\hat{D} = (\hat{d}_{i,j}) \in \mathbb{R}^{N+1,N+1}$ with

$$\hat{d}_{i,j} \;=\; \begin{cases} \frac{2j}{c_i},\ j = i+1, i+3, \ldots, N \\ 0\,, \qquad\qquad \text{else} \end{cases}$$

and

$$c_i \;=\; \begin{cases} 2,\ i = 0, \\ 1,\ \text{else}. \end{cases}$$

Now we are able to write the spectral derivative matrix D for the first derivative. It is explicitly given by

$$D = T\hat{D}T^{-1} \in \mathbb{R}^{N+1,N+1}.$$

The spectral operator can be efficiently evaluated by Fast Fourier Transforms (FFTs) in $O(N \log N)$ arithmetic operations. We further introduce the identity matrix $I \in \mathbb{R}^{N+1,N+1}$. By tensor product representation

$$A \otimes B = (Ab_{i,j})_{i,j}$$

we are now able to write the spectral partial derivatives:

$$\frac{\partial}{\partial x} \cong D_x = D \otimes I, \ \frac{\partial}{\partial y} \cong D_y = I \otimes D.$$

The partial derivatives for a general rectangular element with the step size h_x resp. h_y in the x- resp. y-direction are now given by $\frac{2}{h_x} D_x$ resp. $\frac{2}{h_y} D_y$.

4 The Overlapping Schwarz Method

We first introduce the overlapping mesh based on an equidistant grid. Let k_d denote the number of elements in one direction and the equidistant step size h is then given by $h = 1/k_d$. The equidistant elements are stretched by $\delta = \alpha h, \ \alpha > 0$ on both sides. In the practical computations we choose $\alpha = 0.1$. Now by using the equidistant nodes $t_i = ih$ we are able to write the left and right interval bounds. The first interval bounds are given by $l_1 = 0, \quad r_1 = t_1 + \delta$. Then we have

$$l_i \ = \ t_{i-1} - \delta, \quad r_i \ = \ t_i + \delta, \quad i = 2, \ldots, k_d - 1 \tag{6}$$

and finally the bounds of the last interval are $l_{k_d} = t_{k_d-1} - \delta, \quad r_{k_d} = 1$. In the 1D case we obtain the overlapping intervals $[l_i, r_i], \ i = 1, \ldots, k_d$ with an overlapping size of 2δ. In the 2D case we obtain the overlapping rectangular subdomains

$$\Omega_{i,j} \ = \ [l_i, r_i] \times [l_j, r_j] \,, \quad i, j = 1, \ldots, k_d.$$

During the Schwarz iteration one calculates approximations by using interpolated values from the neighbouring subdomains. We once more describe the treatment in the 1D case. The length of the elements near the boundaries are $h + \delta$ whereas the inner elements have a length of $h + 2\delta$. First we interpolate the approximation of the element by means of the inverse Chebyshev transform with T^{-1} and then compute the Chebyshev expansion (on $[-1, 1]$) in the transformed interface points

$$x_l = 2\frac{2\delta}{h+\delta} - 1, \quad x_r = 2\frac{h-\delta}{h+\delta} - 1$$

for boundary elements and in

$$x_l = 2\frac{2\delta}{h+2\delta} - 1, \quad x_r = 2\frac{h}{h+2\delta} - 1$$

for inner elements. In the 2D case the same treatment has to be performed in both directions. For the lid-driven cavity flow problem singularities occur near the four corners. Hence it is useful to choose a mesh which is more dense near the boundaries. As proposed in [25] we also choose a mesh based on the Chebyshev collocation points. The overlapping Chebyshev mesh can be written as above where in (6) the equidistant nodes t_i have to be replaced by the onto $[0, 1]$ transformed Chebyshev nodes

$$c_i = \frac{1}{2}\left(1 - \cos \pi t_i\right), \quad i = 1, \ldots, k_d. \tag{7}$$

The overlapping size δ is chosen based on the smallest step size, i.e., $\delta = \alpha c_1$. The overlapping equidistant and Chebyshev grids for $k_d = 8$ are plotted in Fig. I. For an efficient parallel implementation we recommend to solve the the subdomain problems in a checkerboard manner. We first solve Dirichlet problems on the subdomains $\Omega_{i,j}\ , i + j\ even$ and then for the $\Omega_{i,j}\ , i + j\ odd$. On each subdomain we use the above spectral collocation scheme. The Navier-Stokes system yields $4(N + 1)^2$ conditions of collocation for the four unknown functions u, v, ω, p. Furthermore for the Schwarz method we have Dirichlet boundary conditions in $4N$ boundary points for the velocity components u, v. This leads to an overdetermined system of $4(N + 1)^2 + 8N$ equations for $4(N + 1)^2$ unknowns. For the corresponding matrix

$$A \in \mathbb{R}^{4(N+1)^2+8N,4(N+1^2)}$$

the linear system $Az = r$ is solved by least-squares in the discrete L^2-norm which leads to the normal equations

$$A^t Az = A^t r.$$

After elimination of the constant mode for the pressure we obtain a symmetric and positive definite system which can be efficiently solved by (preconditioned) conjugate gradient methods.

5 Numerical Results

We consider examples which were also treated by Haschke and Heinrichs [12] for splitting schemes. First we introduce a smooth example where the velocity components u, v and the pressure p are given by

$$
\begin{aligned}
u(x,y) &= \sin\left(\frac{\pi\hat{x}}{2}\right)\cos\left(\frac{\pi\hat{y}}{2}\right) && (8)\\
v(x,y) &= -\cos\left(\frac{\pi\hat{x}}{2}\right)\sin\left(\frac{\pi\hat{y}}{2}\right) && (9)\\
p(x,y) &= \frac{1}{4}\left(\cos\pi\hat{x} + \cos\pi\hat{y}\right) + 10(\hat{x}+\hat{y}) && (10)
\end{aligned}
$$

where $\hat{x} = 2x - 1,\ \hat{y} = 2y - 1$ are the coordinates in $[-1, 1]$. At the begin of the time integration we use some nested Schwarz iteration. We first determine an approximations on a coarse grid with $k_d = 2$ which is interpolated to an intermediate grid with $k_d = 4$. Then we use further Schwarz steps until a certain accuracy is achieved. The approximation for $k_d = 4$ is then interpolated to the finest grid with $k_d = 8$ where further Schwarz iterations lead to good initial guess. Then we start the time integration with the backward Euler Scheme where in each time step only a few (one or two) Schwarz steps are necessary for convergence. For a Reynolds number of $Re = 1/\nu = 100$ we calculated the discrete L^2 and H^1 errors on the overlapped equidistant grid with $k_d = 8$. For increasing N we observe from table I the high spectral accuracy.

Furthermore, we consider the regularized cavity flow (see [3, 12]) where the fluid velocity on the edge $y = 1$ is given by

$$u(x,1) = -16x^2(1-x)^2,\ v(x,1) = 0 \tag{11}$$

where $u = v = 0$ on the other three edges.The source term $\mathbf{f}$ is identical to zero. After the steady state is reached we also calculated the streamfunction ψ by solving the equation

$$\Delta\psi = -\omega \ \text{ in } \Omega = (0,1)^2.$$

In order to compare our results with the results in [3] we calculated the maximal value of ψ on the collocation points of Ω. This value is denoted by M_1. Furthermore we computed the maximal value of ω on the edge $y = 1$. This value is denoted by M_2. In table II we present our results for $Re = 100,\ 400$ and $N = 3, 6$. The numerical results are in good agreement with the results obtained in [3].

Finally we consider the lid-driven cavity flow problem which is less regular because of the boundary conditions

$$u(x,1) = -1, \ v(x,1) = 0. \tag{12}$$

The numerical results ($k_d = 8, N = 8$) were in good agreement with the results obtained by Proot and Gerritsma (see [25]). In each time step only a few Schwarz iterations are employed. The streamfunction and velocity profile for $Re = 1000$ are plotted in fig. II and III.

6 Conclusion

A least-squares spectral collocation scheme for the incompressible Navier-Stokes equations is presented. The problem is iteratively solved by the overlapping Schwarz method. During the time integration only a few Schwarz steps are required for convergence. The solution technique only involves symmetric positive definite linear systems. The method is successfully applied to the regularized and lid-driven cavity flow problem. Numerical simulations confirm the high accuracy of our spectral least-squares scheme.

N	$\|\|u-u^h\|\|_1$	$\|\|v-v^h\|\|_1$	$\|\|p-p^h\|\|$
4	$1.35{\cdot}10^{-2}$	$1.53{\cdot}10^{-2}$	$5.60{\cdot}10^{-4}$
6	$7.17{\cdot}10^{-5}$	$6.76{\cdot}10^{-5}$	$3.10{\cdot}10^{-6}$
8	$4.30{\cdot}10^{-7}$	$4.41{\cdot}10^{-7}$	$2.31{\cdot}10^{-8}$
10	$7.51{\cdot}10^{-9}$	$7.83{\cdot}10^{-9}$	$4.06{\cdot}10^{-9}$

Table I. Numerical results for example (8-10)

	$Re = 100$		$Re = 400$	
N	M_1	M_2	M_1	M_2
3	$8.45{\cdot}10^{-2}$	11.29	$9.54{\cdot}10^{-2}$	18.98
6	$8.36{\cdot}10^{-2}$	13.32	$8.52{\cdot}10^{-2}$	24.72
[3]	$8.34{\cdot}10^{-2}$	13.34	$8.55{\cdot}10^{-2}$	24.78

Table II. Numerical results for the regularized cavity flow (11)

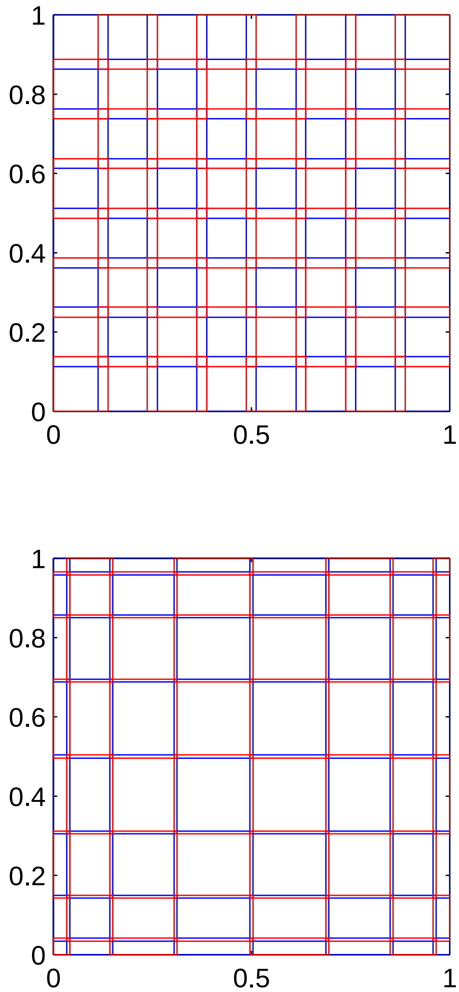

Figure 1: The equidistant and Chebyshev overlapping mesh for $k_d = 8$

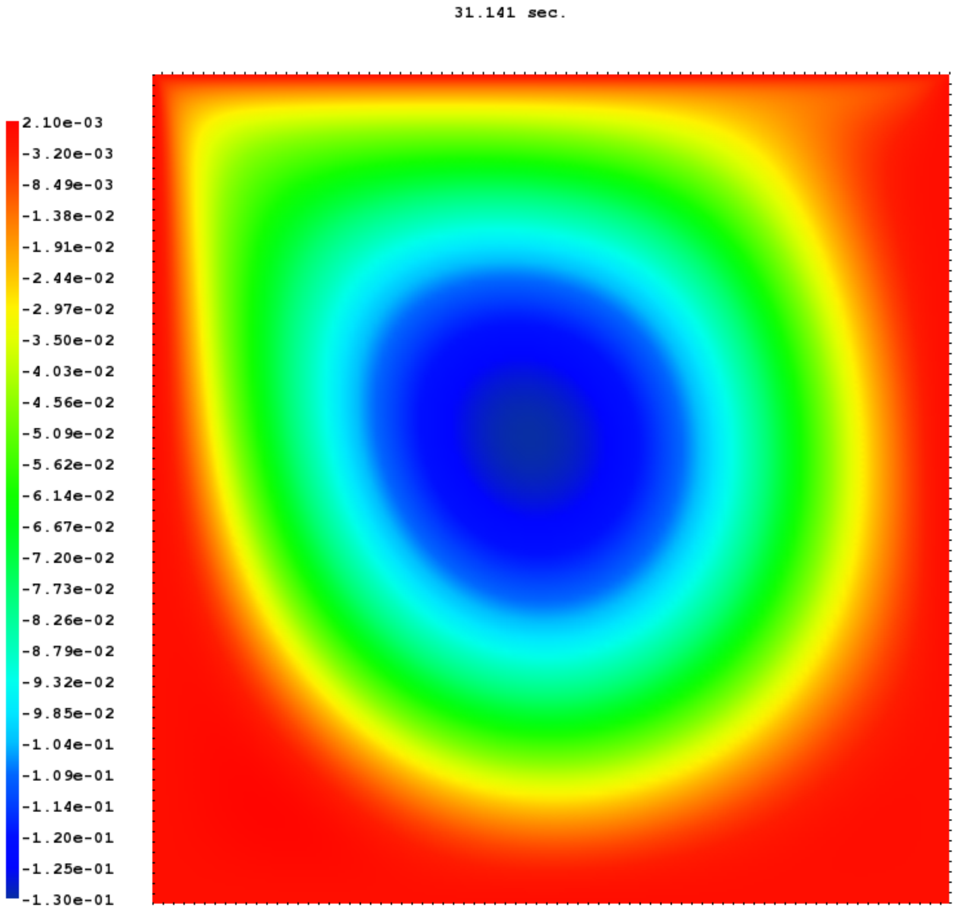

Figure 2: Lid-driven cavity, streamfunction for $Re = 1000$.

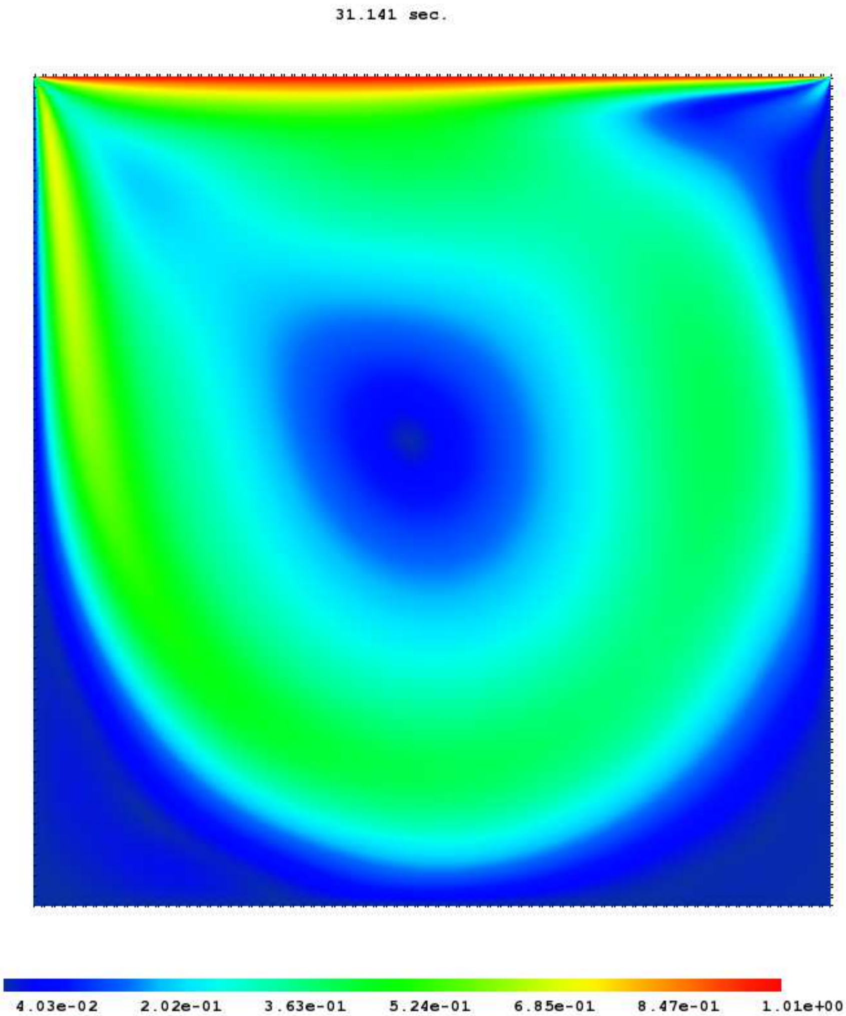

Figure 3: Lid-driven cavity, velocity profile for $Re = 1000$.

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